World of coupled oscillators

Abhijit Sen

Institute for Plasma Research,
Bhat, Gandhinagar 382428, India

WinterSchool: Hands-on Research on Complex Systems
Sao Paulo, July 26-Aug.7, 2009

Theme

- Coupled Oscillators provide a useful paradigm for the study of collective behavior of large complex systems

- A wonderful world to be in – full of interesting mathematical challenges and novel applications – physics, chemistry, biology, economics…..

- Still a very active area of research

- Lends itself easily to \`hands-on style' experiments!

Coupled Oscillators in the Natural World

- Walking, clapping, running…..
- Pacemaker cells in the heart
- Insulin secreting cells in the pancreas
- Neural networks in the brain and spinal cord

  -- control rythmic behaviour like breathing …
- Groups of crickets, frogs in monsoon,
- **Swarms of Fireflies**

A common and striking occurrence is
the emergence of a single rhythm –
“synchrony”

QUESTIONS?

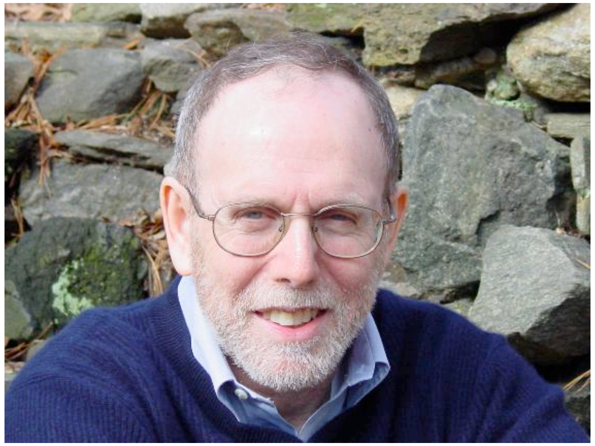
- How do coupled oscillators synchronize?

- Can one construct simple mathematical models to understand this phenomenon?

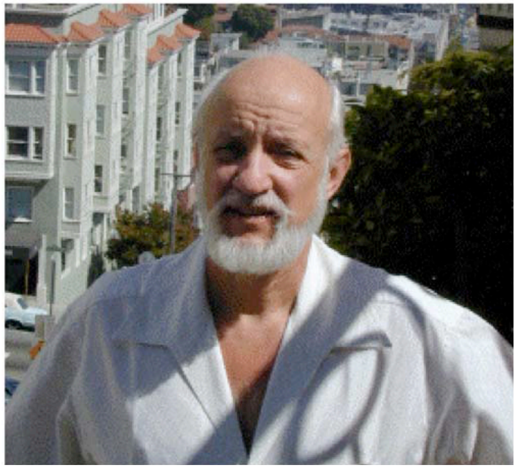
Huygens

~ 1650

Observations and conjectures regarding Pendulum clocks



Charles S. Peskin



Arthur T. Winfree

Mathematical Biologists

Pioneering work around 1970s

- Charles S. Peskin (N.Y.U.) – circa 1975

  - electrical circuit model for pacemaker cells
  - capacitor in parallel with a resistor - constant input current - mimics firing of a pacemaker cell
  - considered an array of identical oscillators - globally coupled (pulse coupling)

TWO CONJECTURES

- System would always eventually synchronize
- It would synchronize even if the oscillators are not quite identical

- **PESKIN** PROVED HIS FIRST CONJECTURE FOR 2 OSCILLATORS

  (ALSO FOUND AN OUT OF PHASE EQUILIBRIUM)

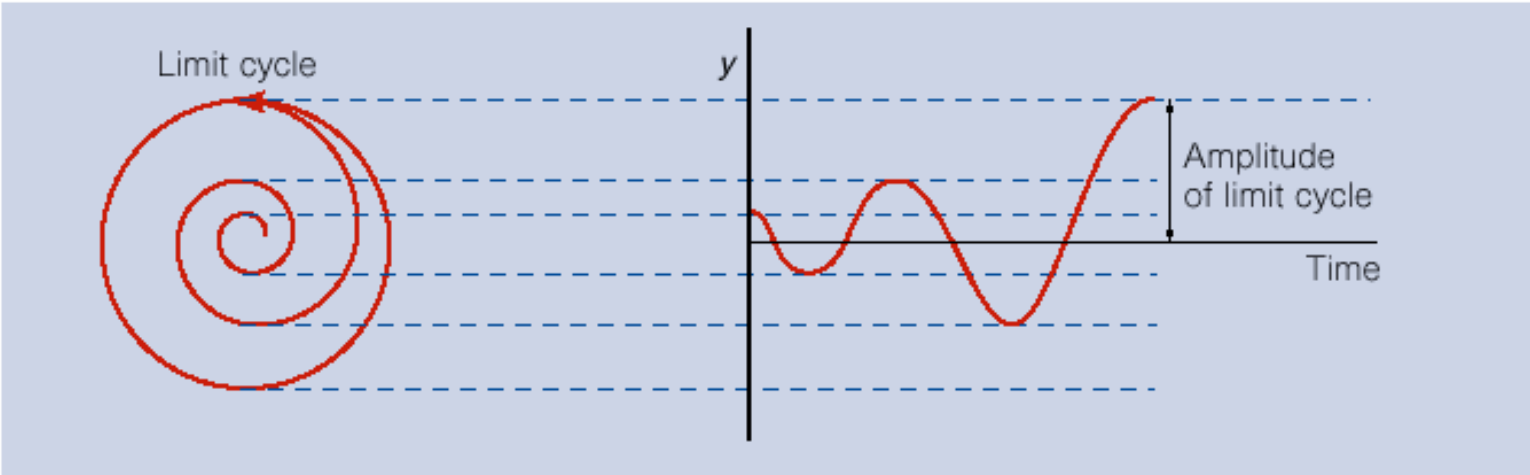
- GENERAL PROOF FOR ARBITRARY NUMBER OF OSCILLATORS WAS OBTAINED 15 YRS LATER (*STROGATZ & MIROLLO*)

- **ARTHUR T. WINFREE** (1966) - graduate student at Princeton

  - MAJOR BREAKTHROUGH
  - CONSIDERED SYSTEM OF COUPLED ***LIMIT CYCLE OSCILLATORS***
  - WEAK COUPLING APPROXIMATION
  - CONSIDERED ONLY PHASE VARIATIONS
  - GLOBAL COUPLING

- **Y. Kuramoto – developed the model further and made extensive use of it.**

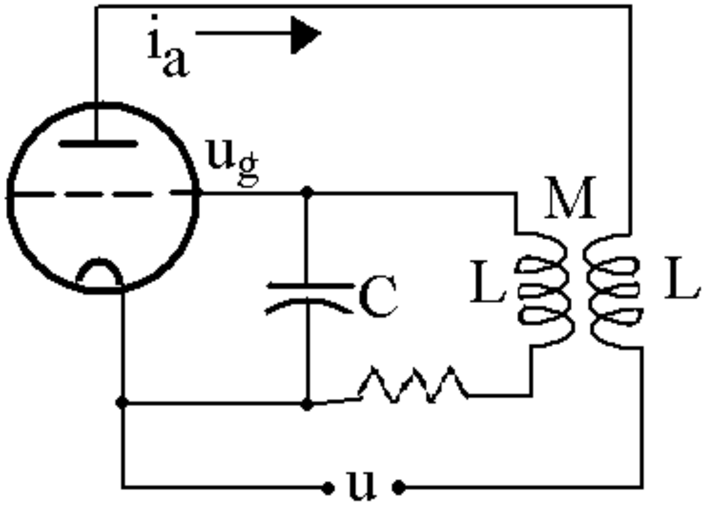
LIMIT CYCLE OSCILLATOR



$$X'' + a\,(X^2 - 1)X' + X = 0, \quad a > 0$$

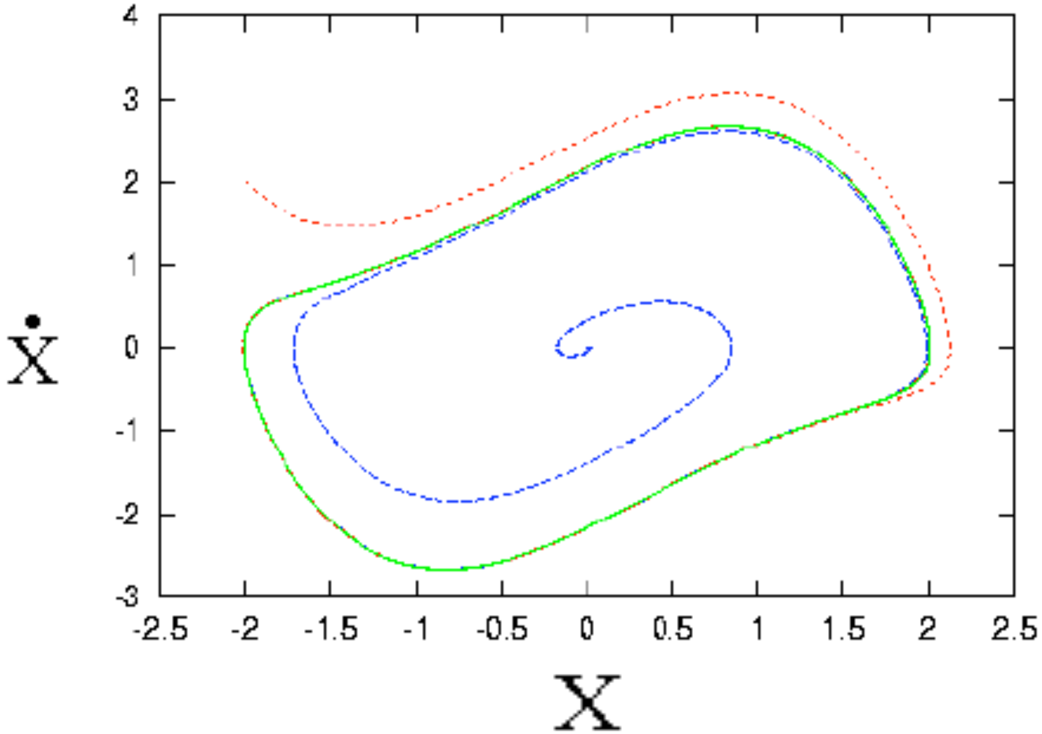
"Isolated closed curve in phase space"

Van der Pol Oscillator



$$i = C\,\frac{d u_g}{dt}$$

$$x = u_g$$



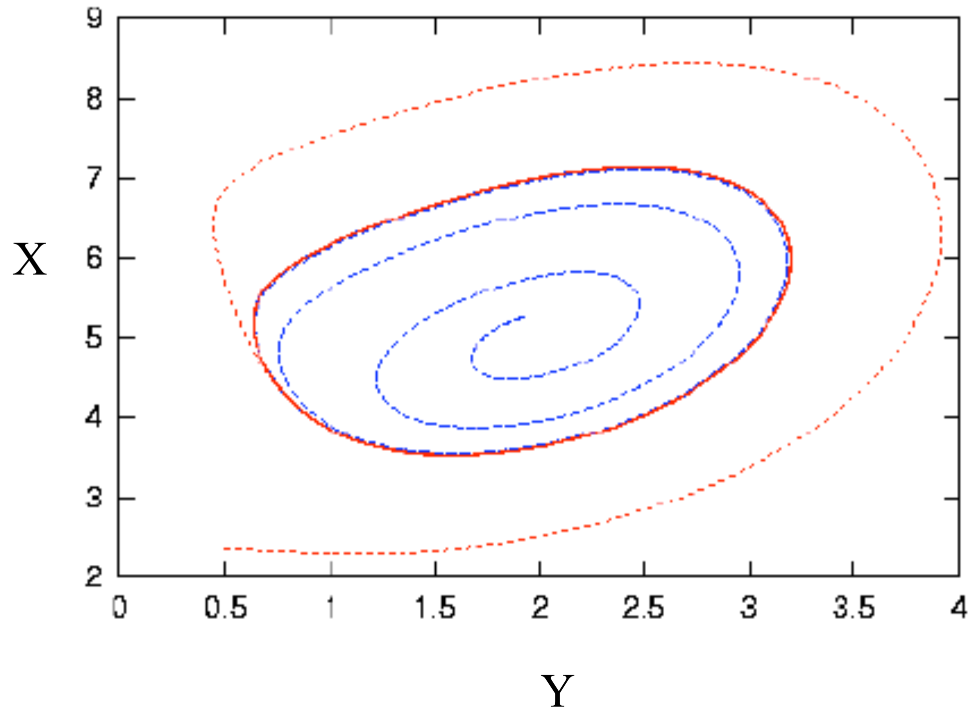
Belousov Zhabotinsky Reaction

Citric acid and bromate ions in a solution of sulfuric acid, and in the presence of a cerium catalyst.

$$\dot{X} = a - X - \frac{4XY}{1+X^2}$$

$$\dot{Y} = bX\left(1 - \frac{Y}{1+X^2}\right)$$

a=10, b=2



A SINGLE HOPF BIFURCATION OSCILLATOR

$$Z^{/}(t) = (a + i\omega - |\,Z(t)\,|^{2})Z(t)$$

$^{/} \rightarrow d/dt$

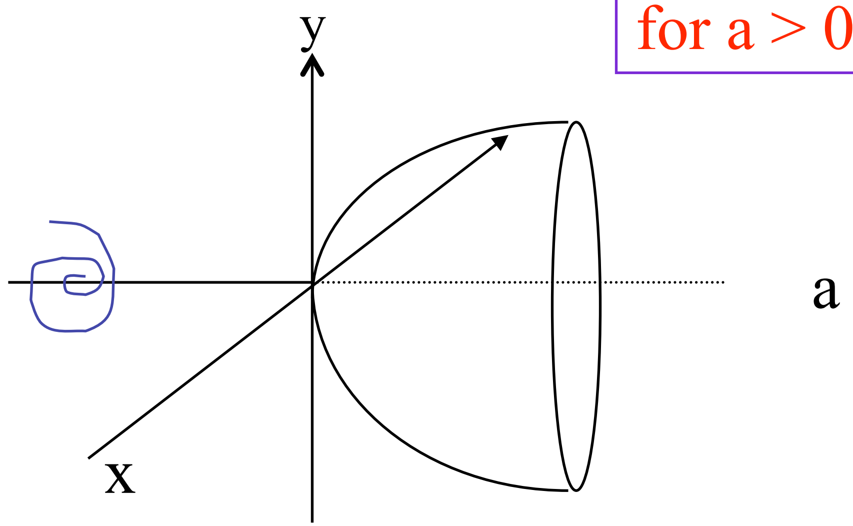
where $Z = X + iY = r\exp(i\theta)$

Stewart – Landau Oscillator

$$\mathbf{r}^{/} = \mathbf{r}\,(\mathbf{a} - \mathbf{r}^{2})$$
$$\theta^{/} = \omega$$

$$\delta r^{/} = a\,\delta r$$

Origin (r=0) stable for a ≤ 0
for a > 0 **limit cycle osc.**



Two Coupled Limit cycle Oscillators

$$\dot{Z}_1(t) = (1 + i\omega_1 - \mid Z_1(t) \mid^2)\, Z_1(t) + K\,[Z_2(t) - Z_1(t)],$$

$$\dot{Z}_2(t) = (1 + i\omega_2 - \mid Z_2(t) \mid^2)\, Z_2(t) + K\,[Z_1(t) - Z_2(t)],$$

K = coupling constant; a=1

In polar coordinates

$$\begin{aligned}
\dot{r}_1 &= r_1(1 - K - r_1^2) + K r_2 \cos[\theta_2 - \theta_1], \\
\dot{r}_2 &= r_2(1 - K - r_2^2) + K r_1 \cos[\theta_1 - \theta_2], \\
\dot{\theta}_1 &= \omega_1 + K \frac{r_2}{r_1} \sin[\theta_2 - \theta_1], \\
\dot{\theta}_2 &= \omega_2 + K \frac{r_1}{r_2} \sin[\theta_1 - \theta_2].
\end{aligned}$$

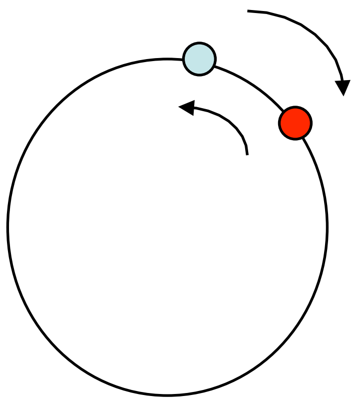
Weak coupling approximation: separation of time scales – short time – relaxation to limit cycle – long time phases interact - - **let $r_1 \approx r_2 \approx$ constant**

Identical oscillators : $\omega_1 = \omega_2$

define $\phi = \theta_2 - \theta_1$

$$\boxed{\dot{\phi} = -2K sin(\phi)}$$

Force tries to reduce phase difference



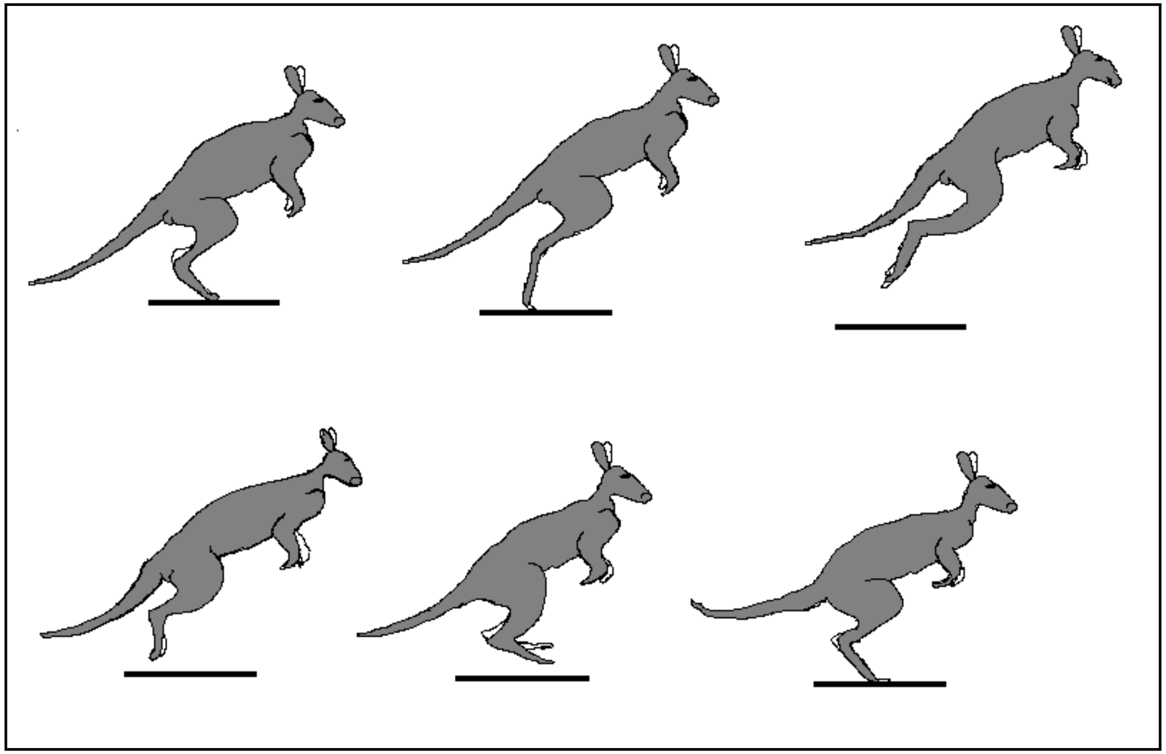
EQUILIBRIA

$\varphi = 0 \Rightarrow \theta_1 = \theta_2$ symmetric state

$\varphi = \pi \Rightarrow \theta_1 = \theta_2 + \pi$ anti -symmetric state

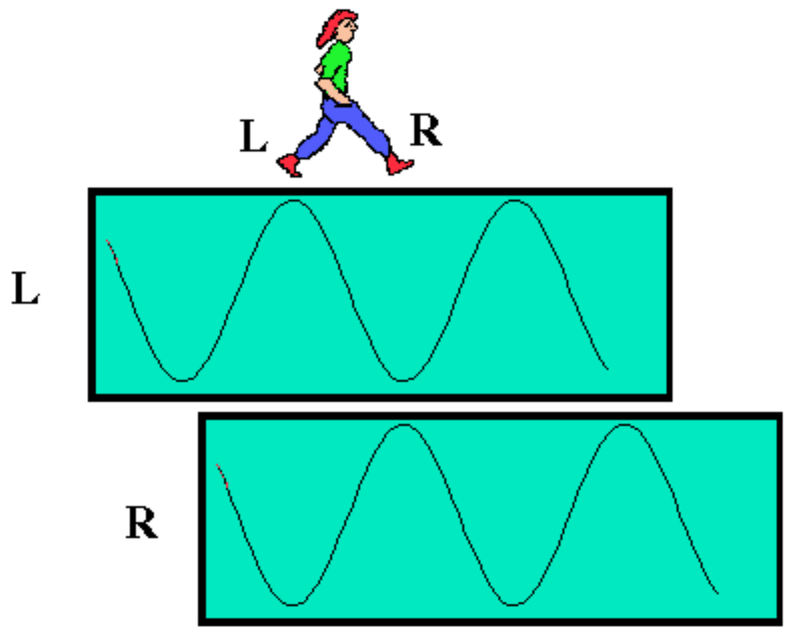
PHASE LOCKING - "synchrony" is only a part of the story - "symmetry breaking" - general scenario

PHASE EQUILIBRIA and ANIMAL GAITS



In Phase

Out of Phase Synchronization in Human Walking / Running



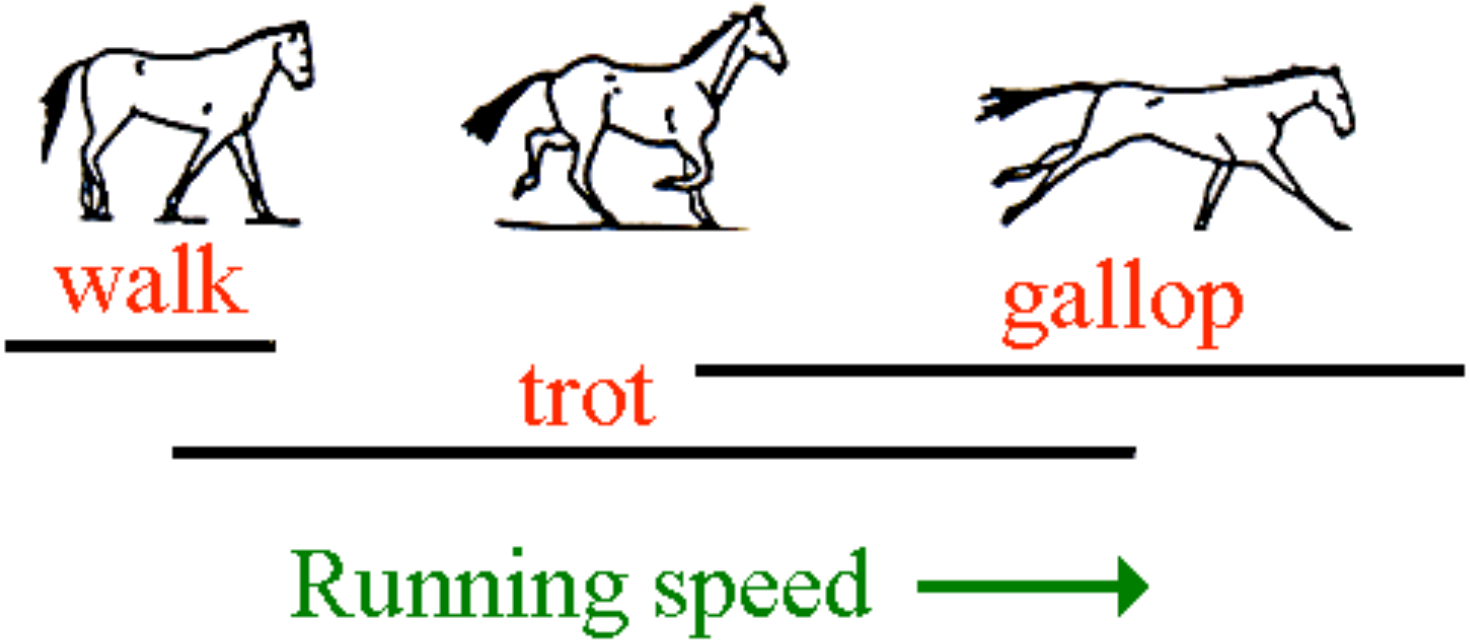
4 OSCILLATORS

$\theta_1 = \theta_2$; $\theta_3 = \theta_4$; $\theta_1 = \theta_3 + \pi$ -- rabbit, camel, horse

$\theta_2 = \theta_1 + \pi/4$; $\theta_3 = \theta_2 + \pi/4$; $\theta_4 = \theta_3 + \pi/4$; -elephant

$\theta_1 = \theta_2 = \theta_3 = \theta_4$ -- GAZELLE

HORSE GAITS



Three Oscillators

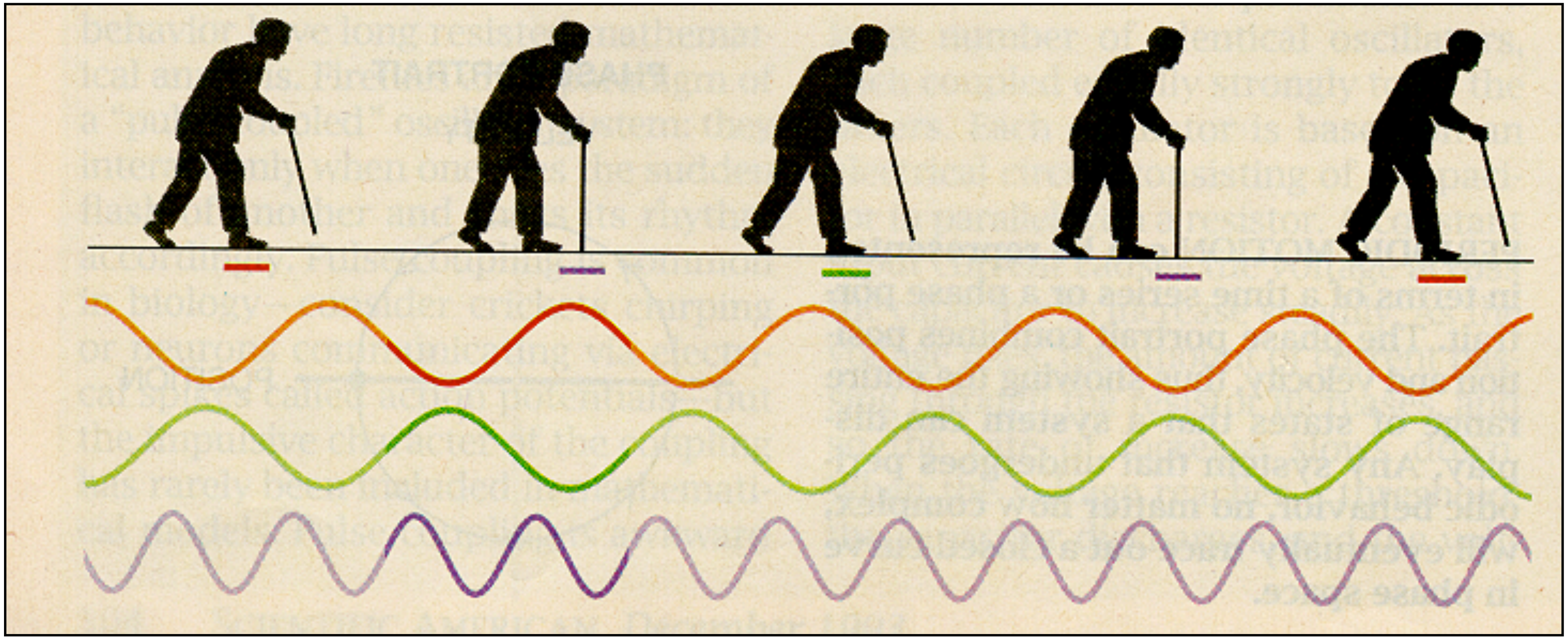
$\theta_1 = \theta_2 = \theta_3$

$\theta_1 = \theta_2 + \pi/3$; $\theta_2 = \theta_3 + \pi/3$;

$\theta_1 = \theta_2$; θ_3 no relation - same frequency

$\theta_1 = \theta_2 + \pi$; θ_3 has twice the frequency

Two out of synchrony and one twice as fast



- 6 OSCILLATORS -- INSECTS, COCKROACHES ETC.

- CENTIPEDE! – traveling wave

Courtesy: Dan Goldman

QUESTION: Coupled osc. Equilibria and Animal gaits - is this a mere coincidence or is there a deeper connection?

- Active area of research
- Central pattern generators (brain and spine)
- Group theoretic methods coupled with generalized Hopf bifurcations
- Clinical experiments

2 NON-IDENTICAL OSCILLATORS

$$\boxed{\dot{\phi} = \Delta - 2K sin(\phi)}$$

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + K\frac{r_2}{r_1}\sin[\theta_2 - \theta_1], \\ \dot{\theta}_2 &= \omega_2 + K\frac{r_1}{r_2}\sin[\theta_1 - \theta_2]. \end{aligned}$$

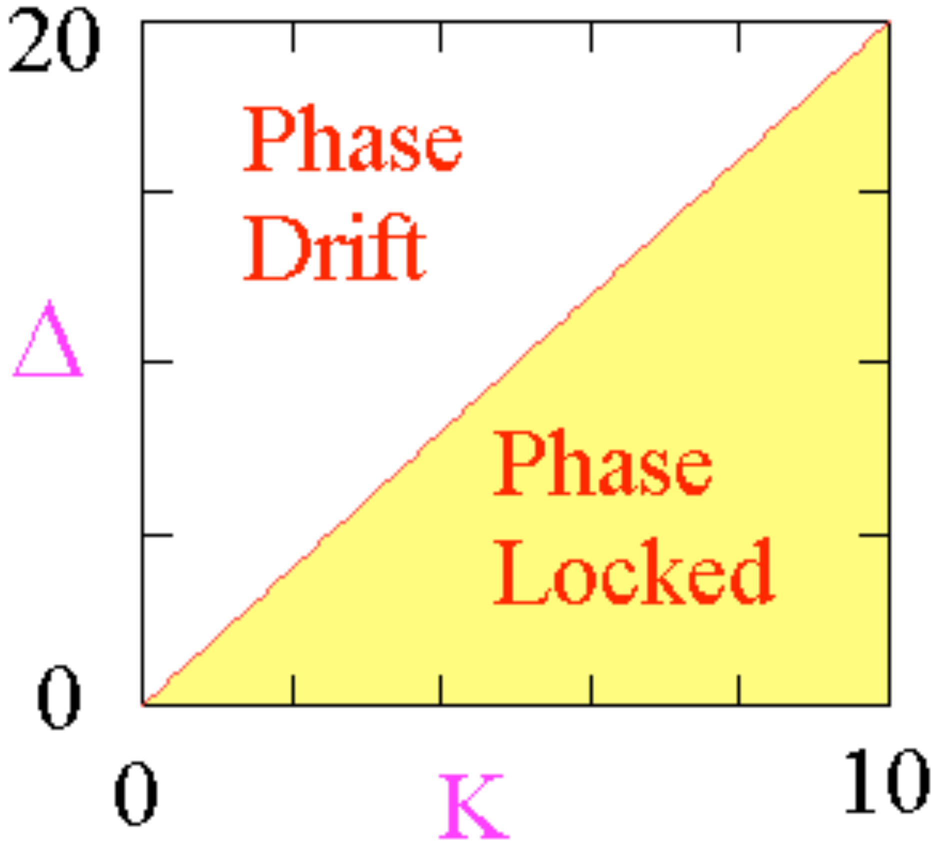
where $\Delta = | \omega_1 - \omega_2 |$

PHASE LOCKING ONLY IF $\Delta \leq 2K$

Then $\dot{\theta} = < \omega > = \dfrac{(\omega_1 + \omega_2)}{2}$ Common frequency

FREQUENCY ENTRAINMENT

Two Phase Coupled Oscillators



N coupled (phase only) oscillators

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \ldots, N$$

Frequencies given by a unimodal distribution function

$$g(\omega) = g(-\omega)$$

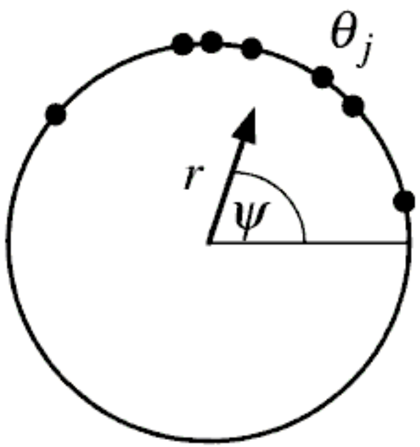
"global coupling" - mean field approximation

Complex order parameter:

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j}$$

$r(t)$ - measure of phase coherence

$\psi(t)$ - average phase



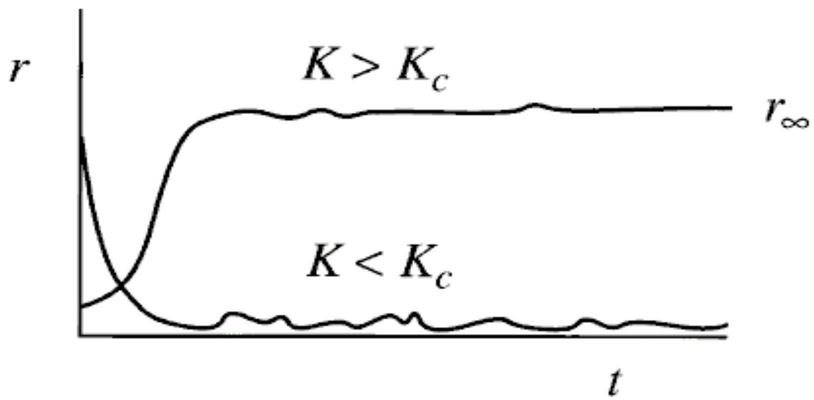
r =1 – synchrony

r = 0 – phase drift

$$\boxed{\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), \quad i = 1, \ldots, N.}$$

Kuramoto solved the equation exactly for r = constant and obtained the threshold condition for synchrony **$K \geq K_c$**

$$\boxed{K_c = \frac{2}{\pi g(0)}}$$

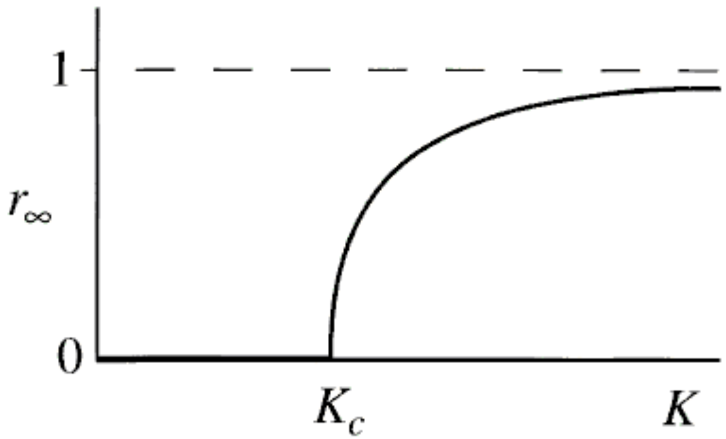


$$r = \sqrt{1 - \frac{K_c}{K}}$$

for

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)}$$

"Second order phase transition"



Near onset

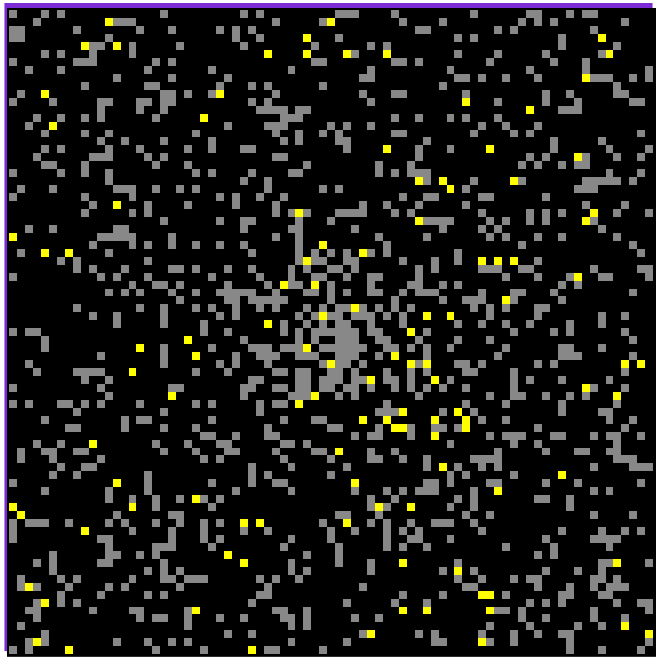
$$r \approx \sqrt{\frac{16}{\pi K_c{}^3}}\sqrt{\frac{\mu}{-g''(0)}}$$

Supercritical bifurcation

for $g''(0) < 0$

(Strogatz and Mirollo, J. Stat. Phys. 63 (1991) 613)

Synchronization in Fire Flies



Synchronization in fireflies

•**S. Strogatz** - *"From Kuramoto to*
•Crawford:exploring the onset of synchronization in populations of coupled oscillators"'
Physica D <u>143</u> (2000) 1-20.

Strong Coupling Limit: Amplitude effects

$$\boxed{\begin{aligned}\dot{Z}_1(t) &= (1 + i\omega_1 - \mid Z_1(t) \mid^2)\ Z_1(t) + K\ [Z_2(t) - Z_1(t)],\\ \dot{Z}_2(t) &= (1 + i\omega_2 - \mid Z_2(t) \mid^2)\ Z_2(t) + K\ [Z_1(t) - Z_2(t)],\end{aligned}}$$

$\mid Z_1 \mid = \mid Z_2 \mid = 0$ is an equilibrium solution

Stability of the origin?

$$\boxed{\lambda^2 - 2(a + i\bar{\omega})\lambda + (b_1 + ib_2) + c = 0}$$

$a = 1 - K,\ b_1 = a^2 - \bar{\omega}^2 + \Delta^2/4,\ b_2 = 2a\bar{\omega}$

$c = -K^2$

Origin stable if $Re(\lambda) < 0$

Marginal Stability Curve

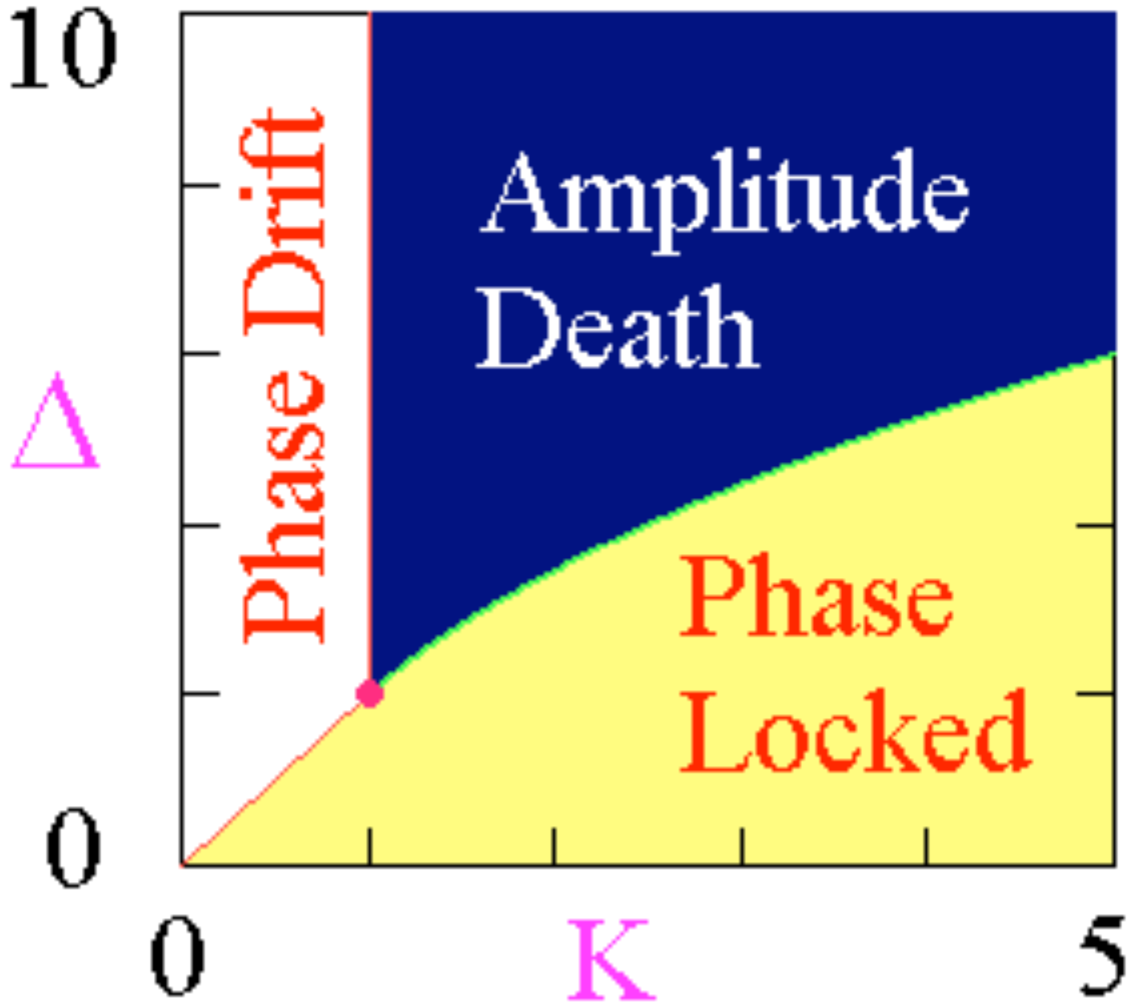
Substitute $\lambda = \alpha + i\beta$ in characteristic equation and

solve it for $\alpha = 0$.

This yields $\boxed{\beta = \bar{\omega}}$

And the conditions: $\boxed{K = 1,\ K = \gamma(\Delta) = \frac{1}{2}(1 + \frac{\Delta^2}{4})}$

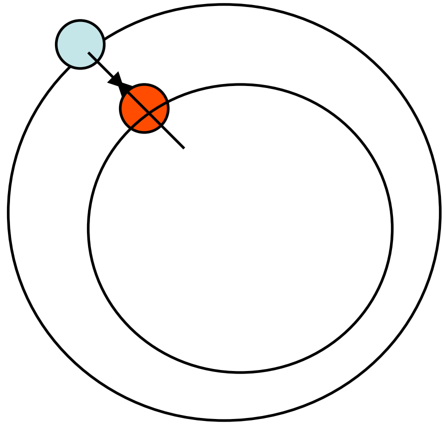
Two Amplitude Coupled Oscillators



Physical picture of amplitude death

(strong coupling limit)

Two oscillators



Each oscillator pulls the other off its limit cycle and they both collapse into the origin $r = 0$ --**AMPLITUDE DEATH**

Happens for K large and Δ large

EXAMPLES OF AMPLITUDE DEATH

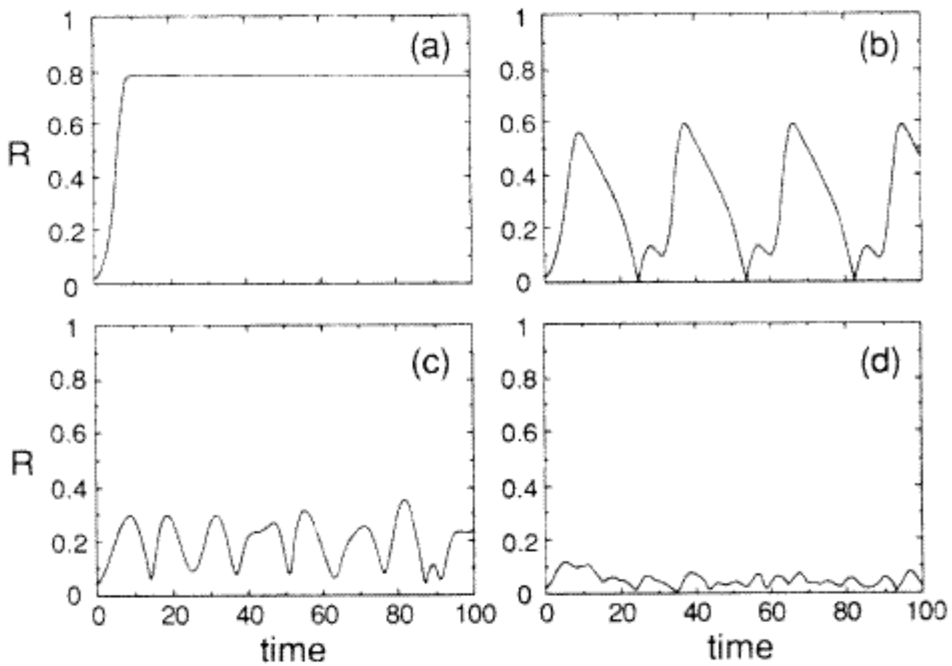
- **CHEMICAL OSCILLATIONS** - BZ REACTIONS
  (coupled stirred tank reactors - Bar Eli effect)

- **POPULATION DYNAMICS**
  Two sites with same predator prey mechanism can have oscillatory behaviour. If species from one site can move to another at appropriate rate (appropriate coupling strength) the two sites may become stable (stop oscillating)

- **ORGAN PIPES**

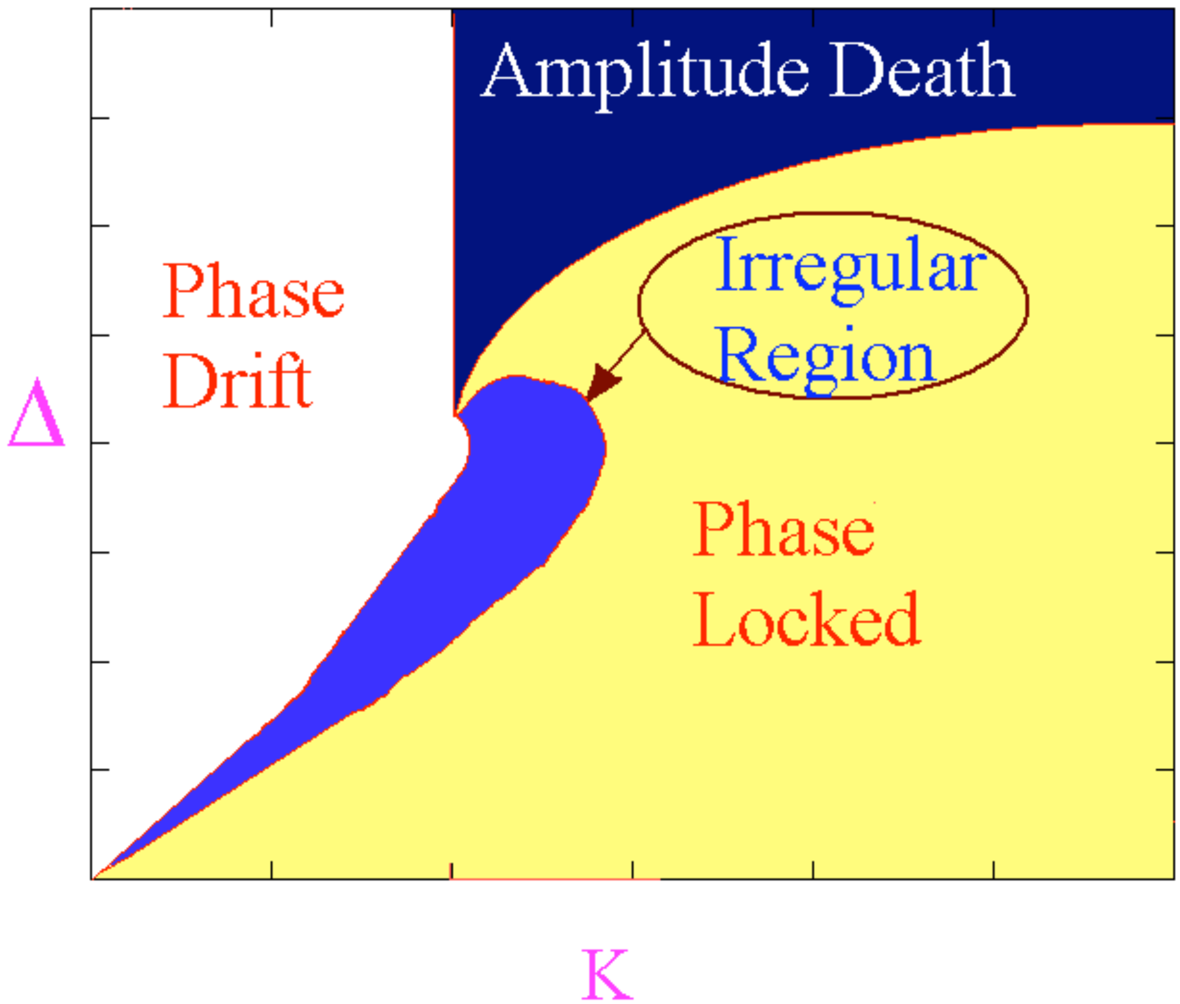
Matthews and Strogatz, PRL 1990

$$\boxed{\dot{z}_j = z_j(1 - |z_j|^2 + i\omega_j) + \frac{K}{N}\sum_{i=1}^{N}(z_i - z_j)}$$



R is the order parameter

Large Number of Amplitude Coupled Oscillators



So far we have looked at systems with “global” coupling
– mean field coupling

What about other forms of coupling?

Short range interactions (nearest neighbour)?

Non-local coupling?

Time delayed coupling

EXTENSION TO SYSTEMS WITH SHORT RANGE INTERACTIONS

- Nearest neighbour coupling
- Limit of very large N – chain of identical oscillators

$$\boxed{\frac{\partial \psi_j}{\partial t} = (1 + i\omega_0 - \mid \psi_j \mid^2)\psi_j + K[\psi_{j+1} - \psi_j] + K[\psi_{j-1} - \psi_j],}$$

In the continuum limit set $\psi_j = \psi(ja)$

Let a → 0 ; ja → x

$$\boxed{\frac{\partial \psi(x,t)}{\partial t} = (1 + i\omega_0 - |\psi(x,t)|^2)\psi(x,t) + K\frac{\partial \psi^2(x,t)}{\partial x^2}}$$

Complex Ginzburg Landau Eqn

Non-local coupling

$$\frac{\partial \psi_j(t)}{\partial t} = (1 + i\omega_0 - \mid \psi_j \mid^2)\psi_j + K \sum_{i=1}^{N} G(j-i)(\psi_i - \psi_j)$$

Continuum limit :

$$\frac{\partial \psi(x,t)}{\partial t} = (1 + i\omega_0 - \mid \psi(x,t) \mid^2)\psi(x,t)$$

$$+ K \int_{-L}^{L} G(x-x') \left[\psi(x',t) - \psi(x,t)\right] dx'$$

Non-local CGLE

Weak coupling limit

$$\psi(x,t) = r(x,t)e^{i\phi(x,t)}$$

Ignore amplitude variations

$$\frac{\partial \phi}{\partial t} = \omega - \int_{-\pi}^{\pi} G(x-x')\sin[\phi(x,t) - \phi(x',t) + \alpha]dx'.$$

``Ring of identical phase oscillators with non-local coupling"

Compare with

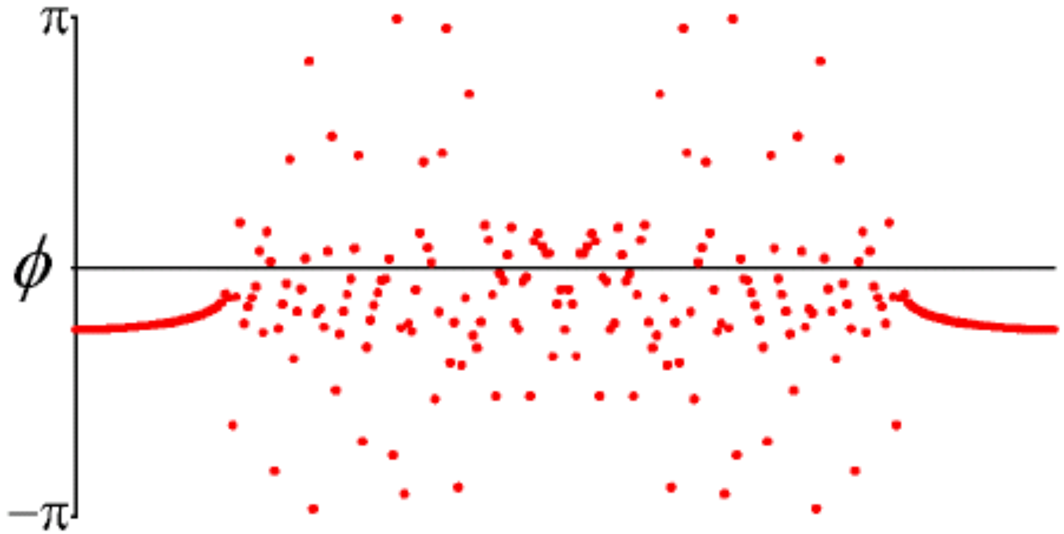
$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

$$G(y) = \frac{\kappa}{2}\exp(-\kappa|y|)$$

Kuramoto and Battogtokh,
Nonlin. Phen. Complex Syst, 5 (2002) 380

“Novel” collective state

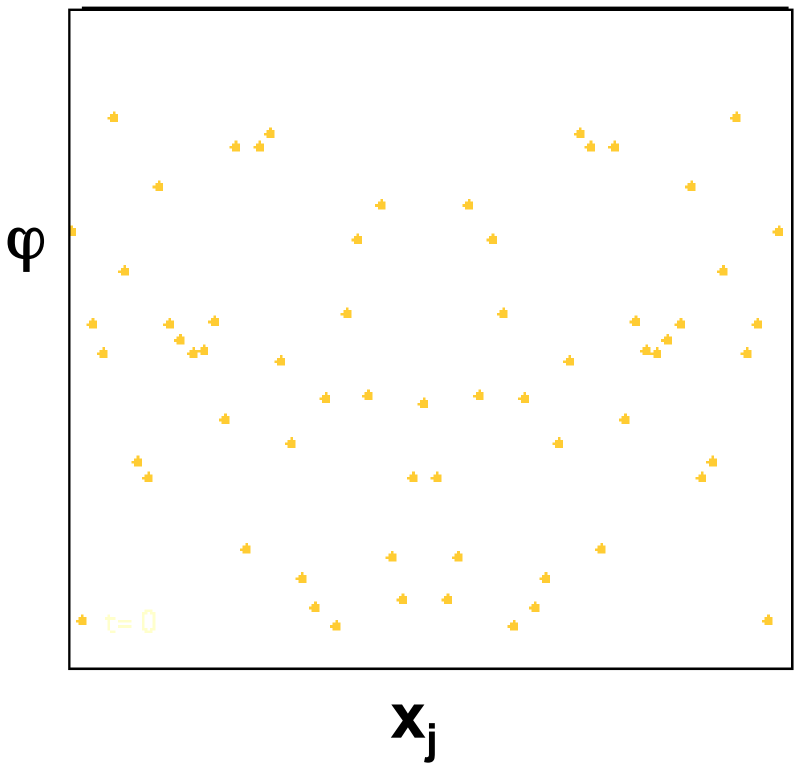
Simultaneous existence of coherent and incoherent states



$\kappa = 4.0, \ \alpha = 1.45, \ N = 256$ oscillators.

“Chimera” **state**

Chimera



Understanding the Chimera state

Define a rotating frame with frequency Ω

Relative phase in that frame $\theta = \phi - \Omega t$

$$R(x,t)e^{i\Theta(x,t)} = \int_{-\pi}^{\pi} G(x-x')e^{i\theta(x',t)}dx'.$$

Complex order parameter

$$\boxed{\frac{\partial \theta}{\partial t} = \omega - \Omega - R\sin\left[\theta - \Theta + \alpha\right]}$$

Look for stationary solutions in which R and Θ are space dependent

$$\omega - \Omega = R(x)\sin\left[\theta^* - \Theta(x) + \alpha\right]$$

Oscillators with $R(x) \geq |\omega - \Omega|$ $\longrightarrow$ Fixed pt. θ^*

$R(x) < |\omega - \Omega|$ drift around the phase circle monotonically.

Time delayed coupling?

Time delay is ubiquitous in real systems due to finite propagation speed of signals, finite reaction times of Chemical reactions, finite response time of synapses etc.

WHAT HAPPENS TO THE COLLECTIVE DYNAMICS OF COUPLED SYSTEMS IN THE PRESENCE OF TIME DELAY?

SIMPLE TIME DELAYED MODEL

$$\boxed{\begin{aligned}\dot{Z}_1(t) &= (1 + i\omega_1 - |Z_1(t)|^2)Z_1(t) + K[Z_2(t-\tau) - Z_1(t)],\\ \dot{Z}_2(t) &= (1 + i\omega_2 - |Z_2(t)|^2)Z_2(t) + K[Z_1(t-\tau) - Z_2(t)],\end{aligned}}$$

(Reddy, Sen and Johnston, Phys. Rev. Letts. 80 (1998) 5109;
Physica D 129 (1999) 15)

Weak coupling limit

$$\dot{\theta}_1 = \omega_1 + K \sin[\theta_2(t-\tau) - \theta_1],$$
$$\dot{\theta}_2 = \omega_2 + K \sin[\theta_1(t-\tau) - \theta_2].$$

Phase locked solution: $$\dot{\theta}_j = \Omega t + \alpha$$

$$\Omega = \bar{\omega} - 2K sin(\Omega\tau)$$

$$\boxed{\bar{\omega} = \frac{\omega_1 + \omega_2}{2}}$$

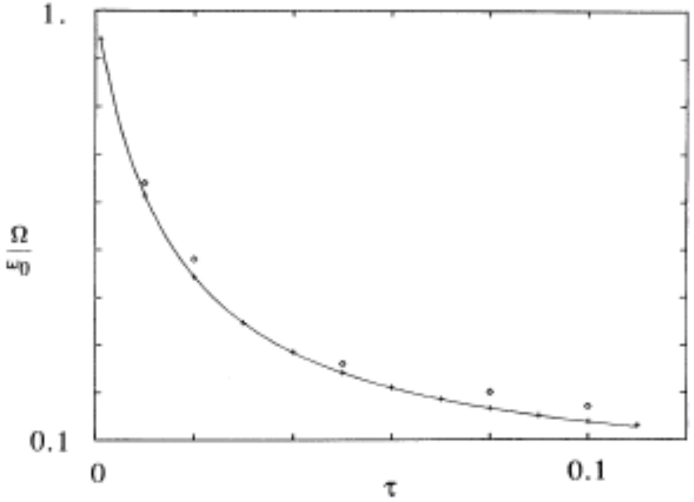
- ***Multiple frequency states***
- ***Frequency suppression***

$$\Omega \approx \frac{\bar{\omega}}{1 + 2K\tau}$$

Early work by Schuster et al

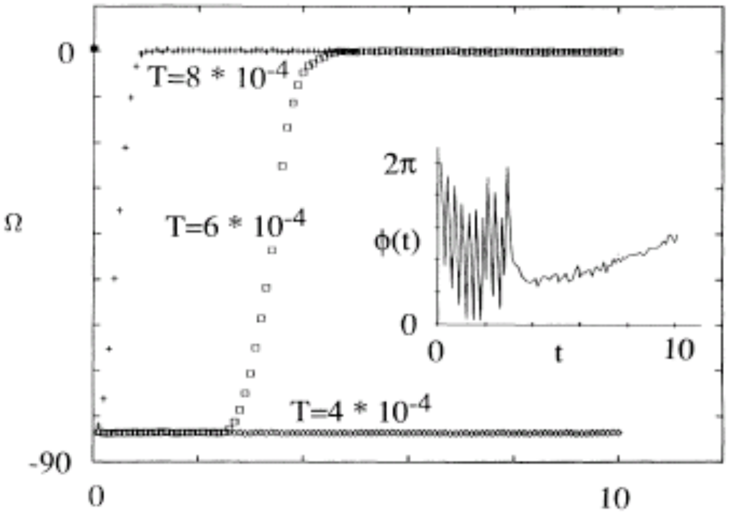
- **Multiple Frequencies:**

  H.G. Schuster and P. Wagner,
  Prog. Theor. Phys. 81 (1989) 93



- **Frequency suppression**

Niebuhr, Schuster & Kammen,
Phys.Rev.Lett. 67 (1991) 2753



Kuramoto model

$$\frac{d\phi_i(t)}{dt} = \omega_0 + K\sum_j \sin[\phi_j(t-\tau) - \phi_i(t)] + \eta_i$$

Strong Coupling Limit:

Linear stability analysis of the origin Z=0

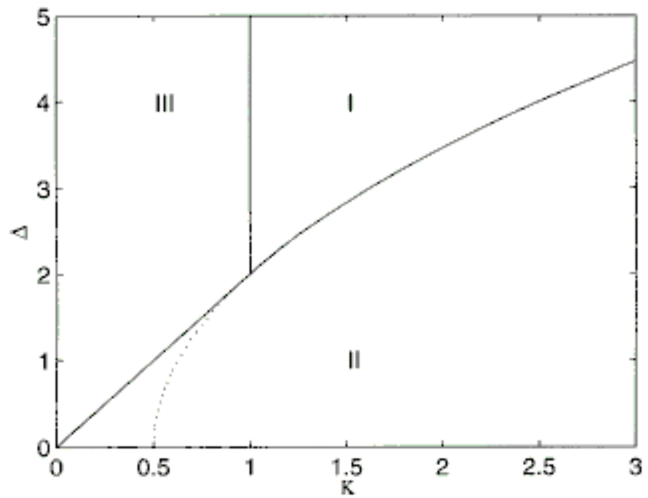
Amplitude Death

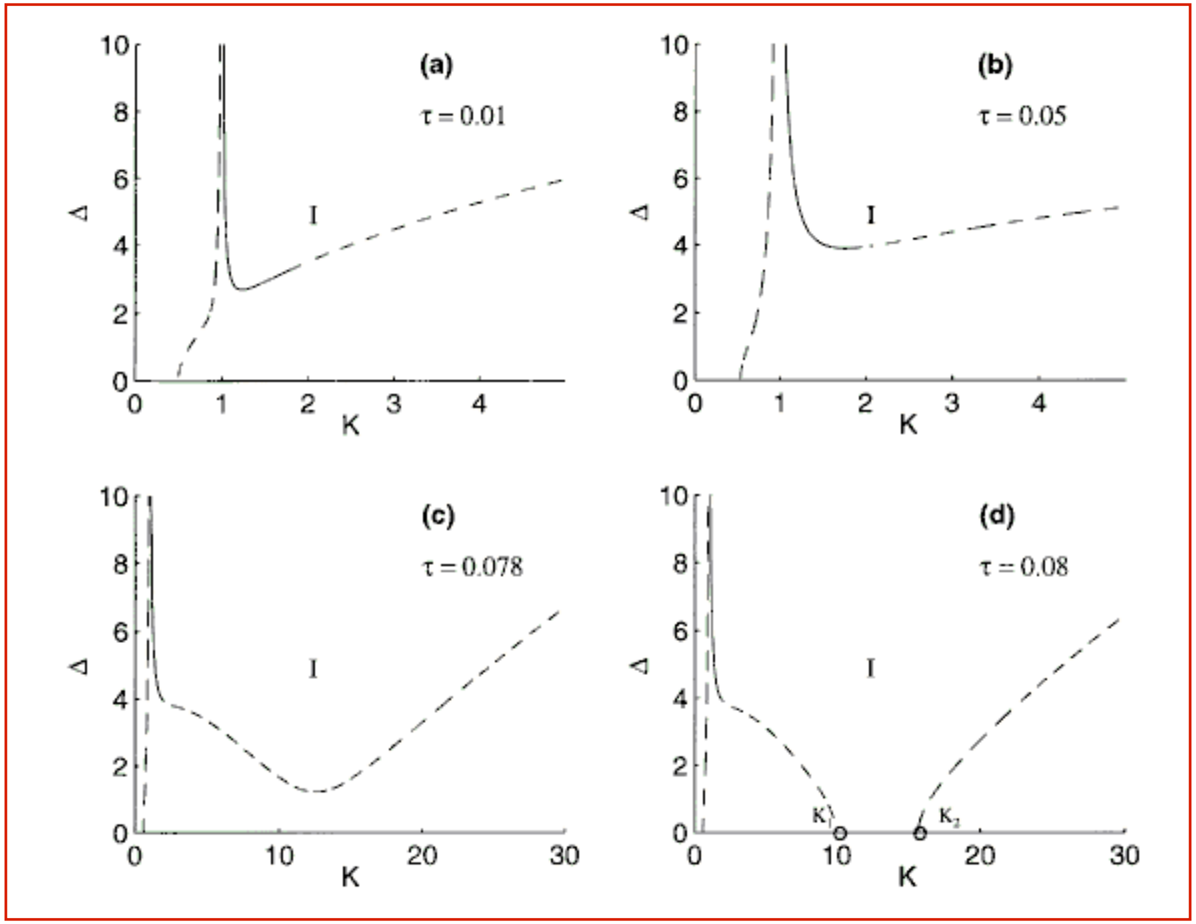
Eigenvalue equation:

$$\boxed{(a - \lambda + i\omega_1)(a - \lambda + i\omega_2) - K^2 e^{-2\lambda\tau} = 0}$$

$$a = 1 - K$$

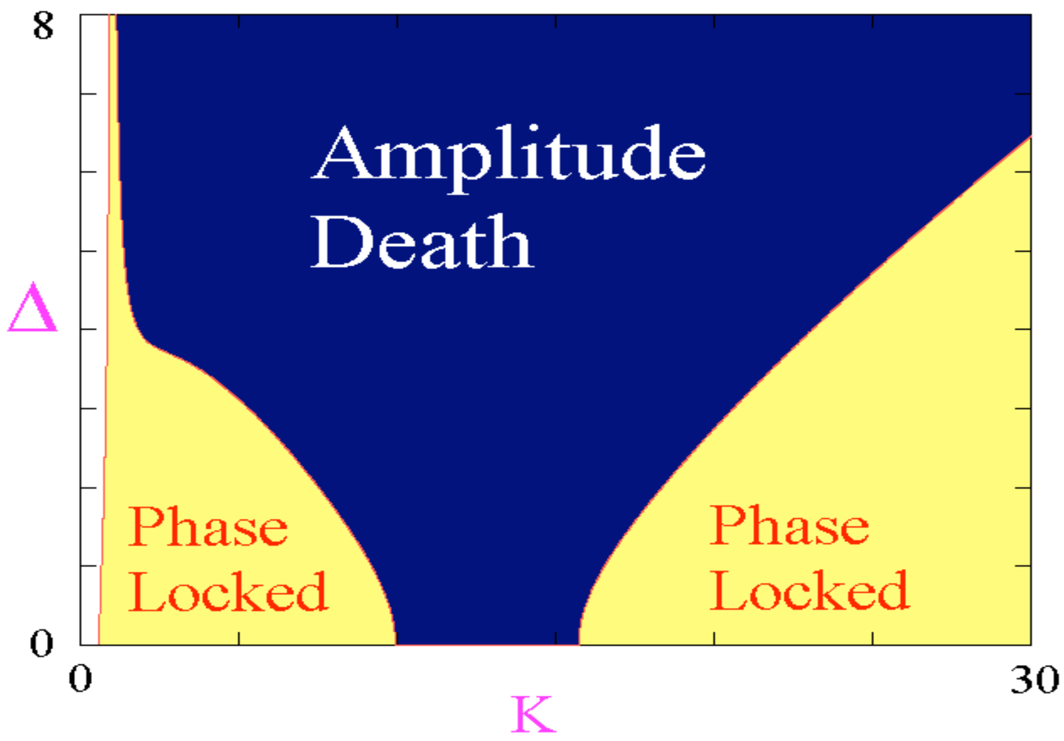
For τ = 0 detailed analysis by
D.G. Aronson, G.B. Ermentrout and
N. Kopell, Physics **41 D** (1990) 403

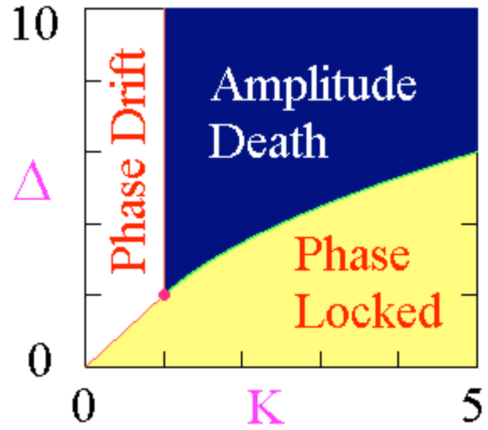




(a)
τ = 0.01
(b)
τ = 0.05
(c)
τ = 0.078
(d)
τ = 0.08
Δ
K
I
K1
K2

Two Coupled Oscillators with Delay



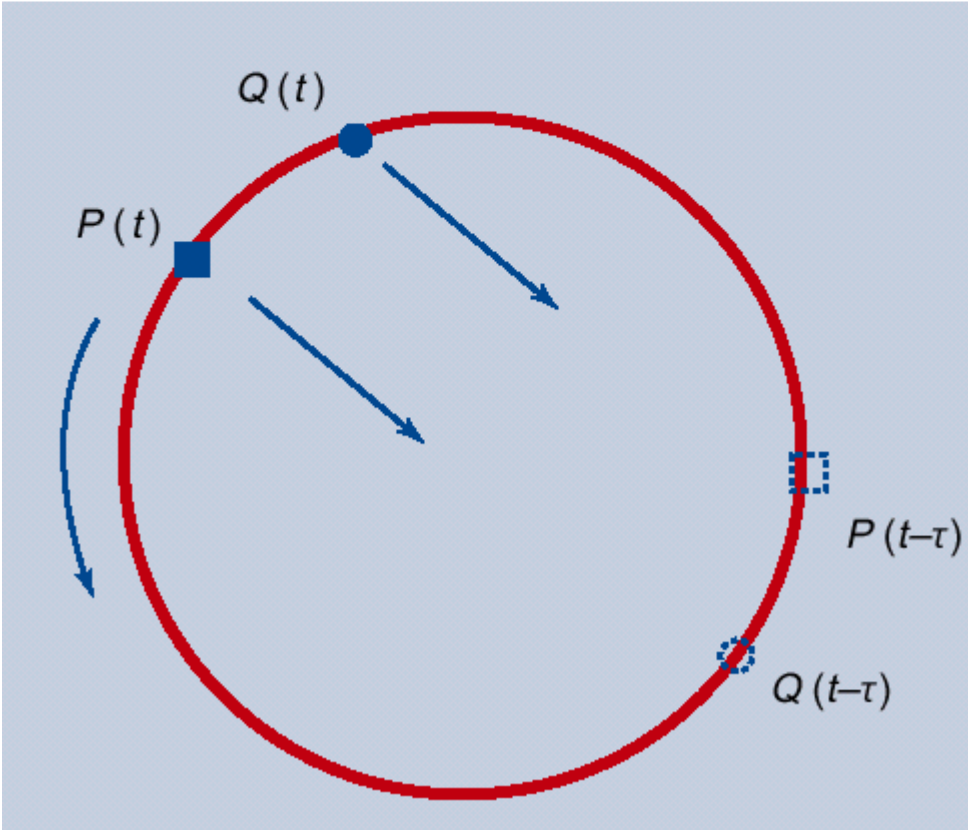


(no delay)

Identical Oscillators can DIE!

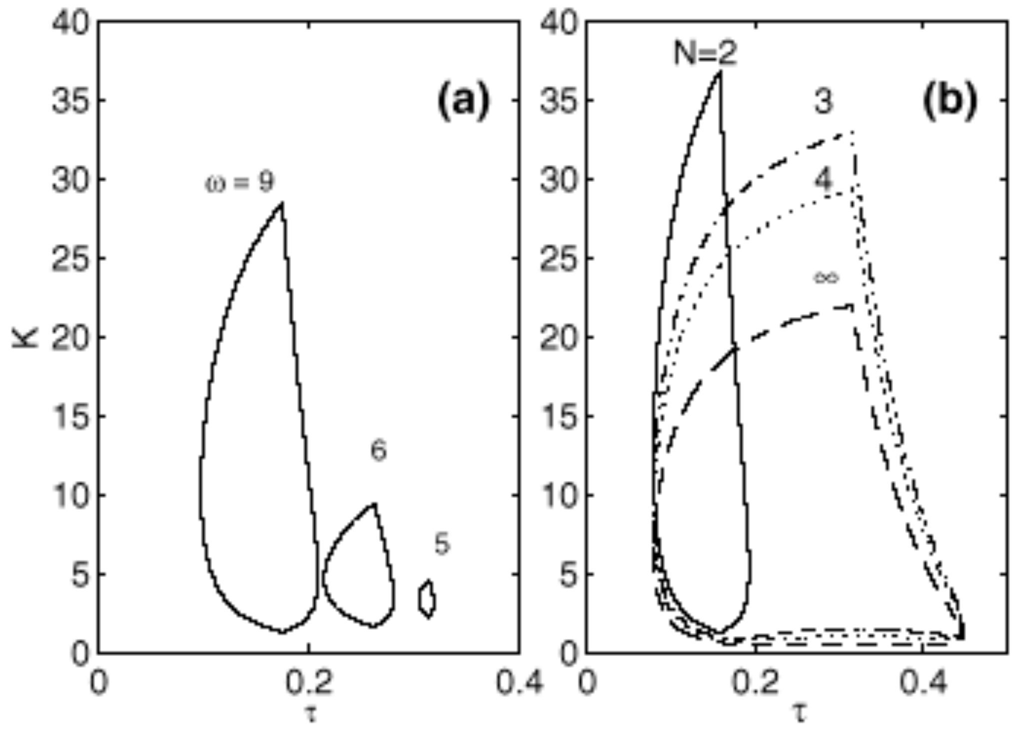
(Reddy, Sen and Johnston, Phys. Rev. Letts. 80 (1998) 5109; Physica D 129 (1999) 15)

Geometric Interpretation of delay induced death in identical oscillators



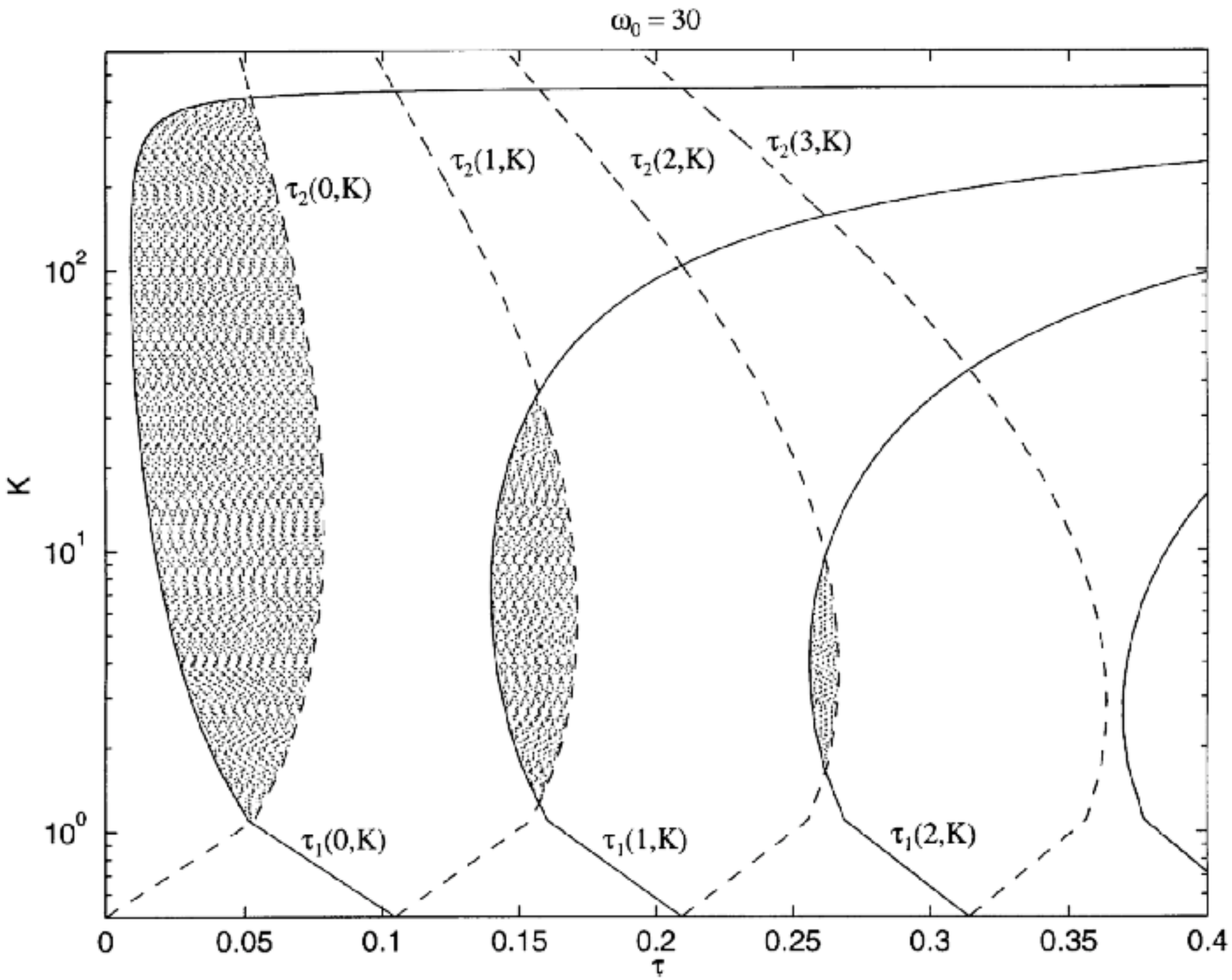
The current state P(t) is pulled towards the retarded state Q(t-τ) of the other oscillator and vice-versa. For appropriate values of K and time delay both oscillations will spiral inwards and die out.

- **Existence of death islands in K - τ space**

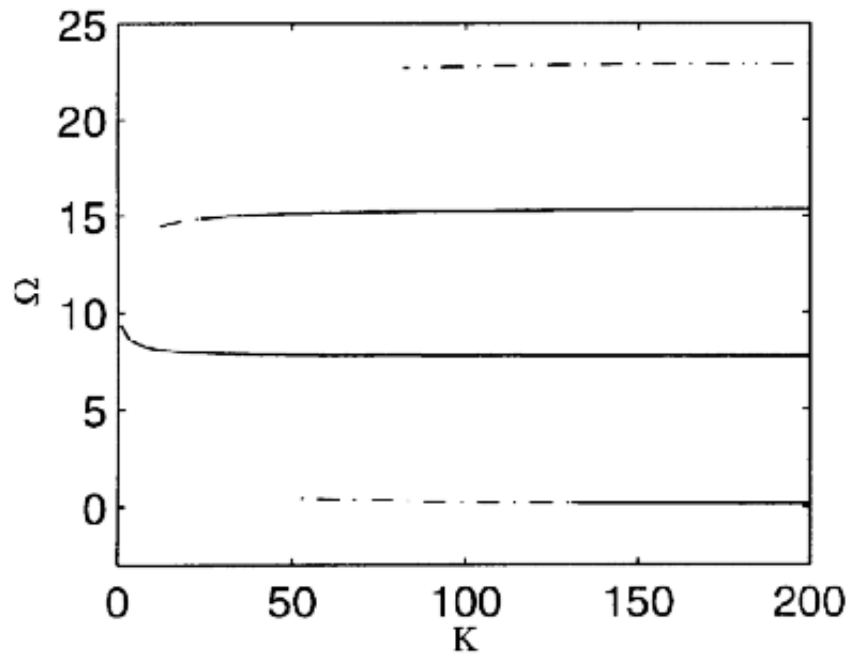


Size, shape vary with N and ω

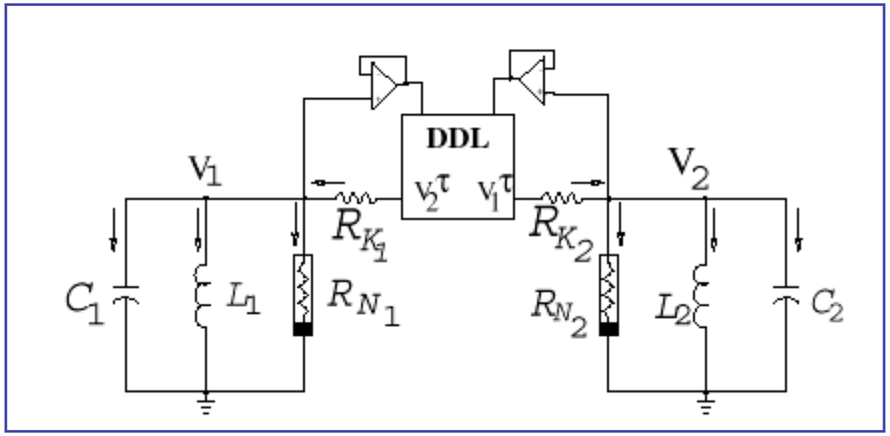
• Existence of multiple death islands



- **Existence of higher frequency states and their stability**



•Experimental verification carried out on coupled nonlinear circuits
(Reddy, Sen, Johnston, PRL, **85** (2000) 3381)



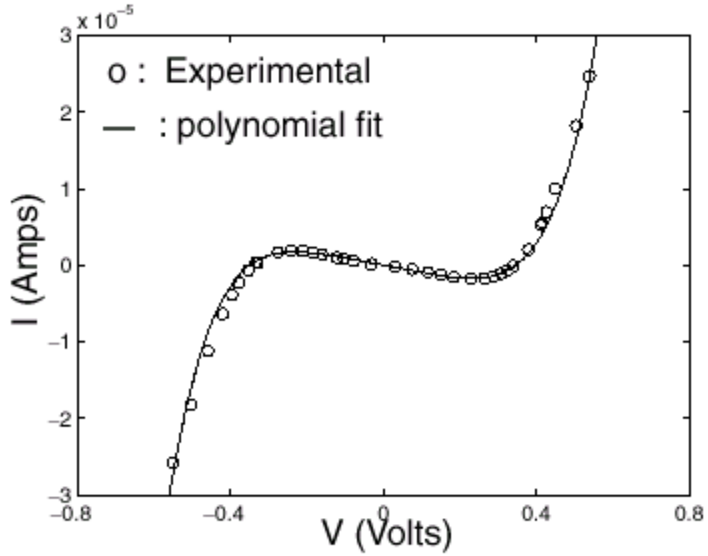
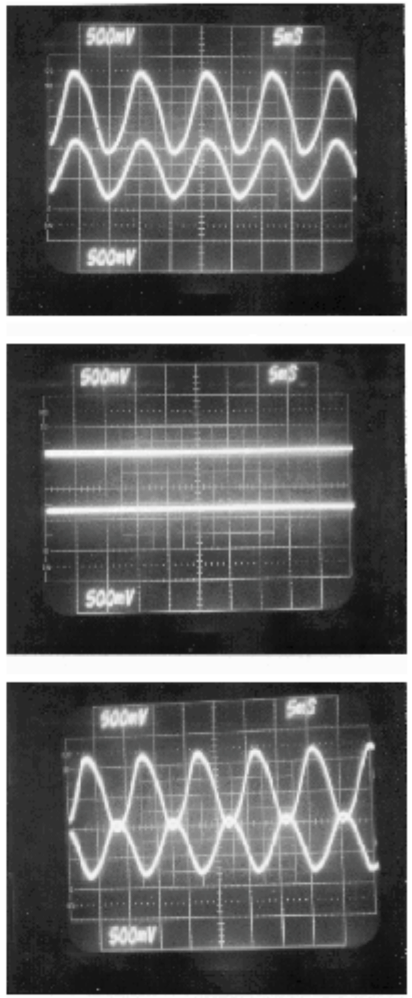


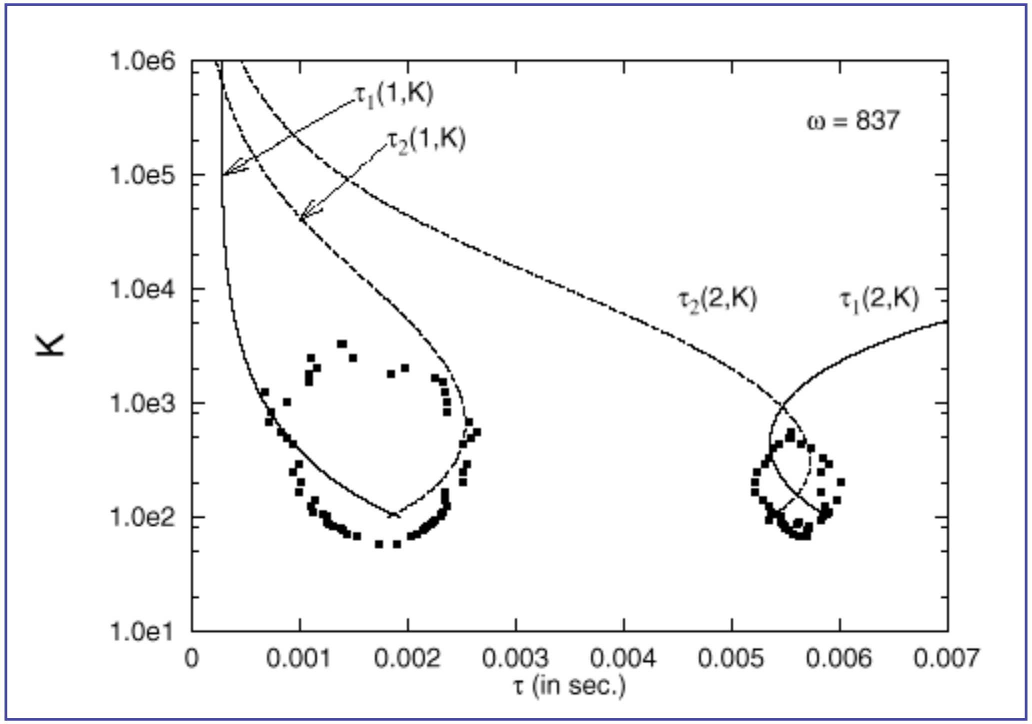
FIG. 2. The V-I characteristics of the nonlinear component R_N. The continuous line is a polynomial fit of the experimental points.

$$\ddot{V}_i + g(V_i)\dot{V}_i + \omega_i^2 V_i = K_i[\dot{V}_j(t-\tau) - \dot{V}_i(t)]$$

- Death state confirmed
- In-phase and out-of-phase oscillations seen



• **Existence of death islands and their multiple connectedness**.



• **IN-PHASE AND ANTI-PHASE LOCKED STATES**

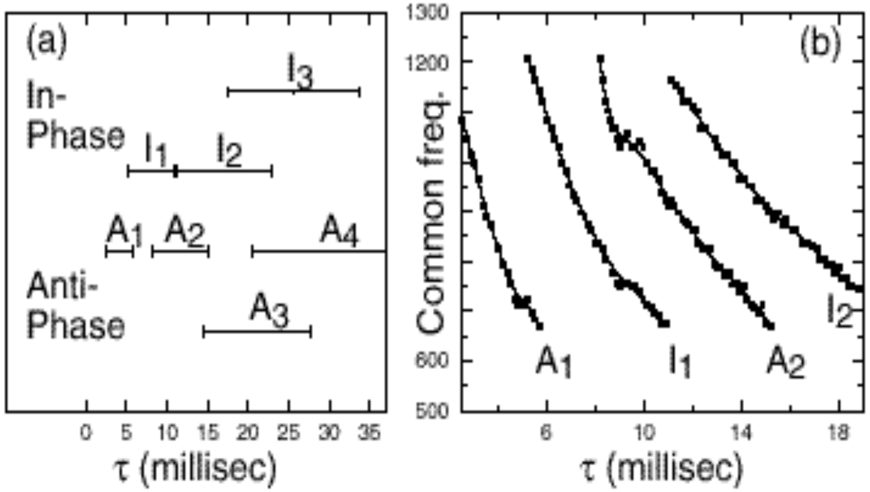


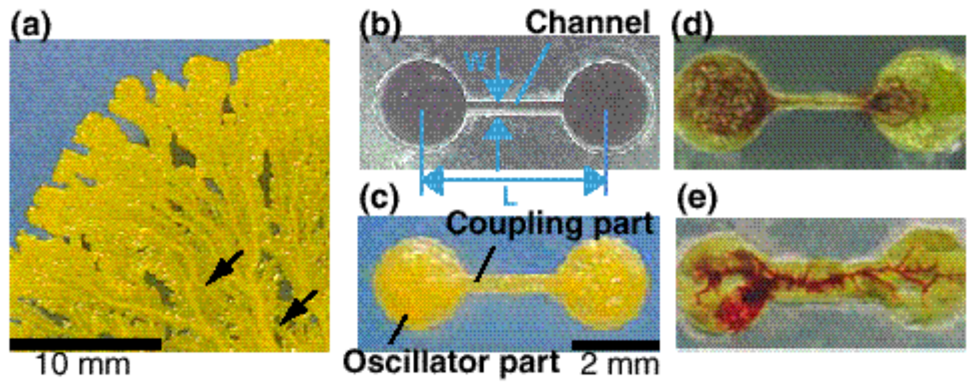
FIG. 6. (a) Coexistence of in-phase- and anti-phase-locked states and (b) suppression of the phase-locked states as τ is increased for $K = 1000\ s^{-1}$ and $\omega = 837\ s^{-1}$.

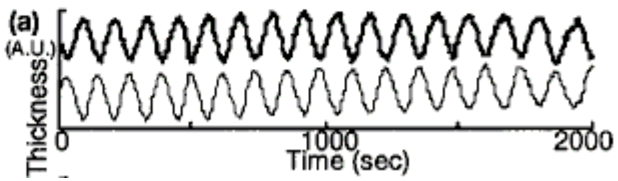
Time delay effects in a living coupled oscillator system

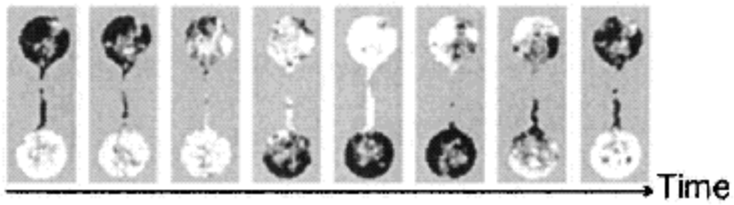
(Takamatsu et al, PRL 85 (2000) 2026)

Experiments with plasmodium of slime mold

- contraction/relaxation states
- time delay and coupling controlled by size of tube
- observed in-phase/anti-phase oscillations



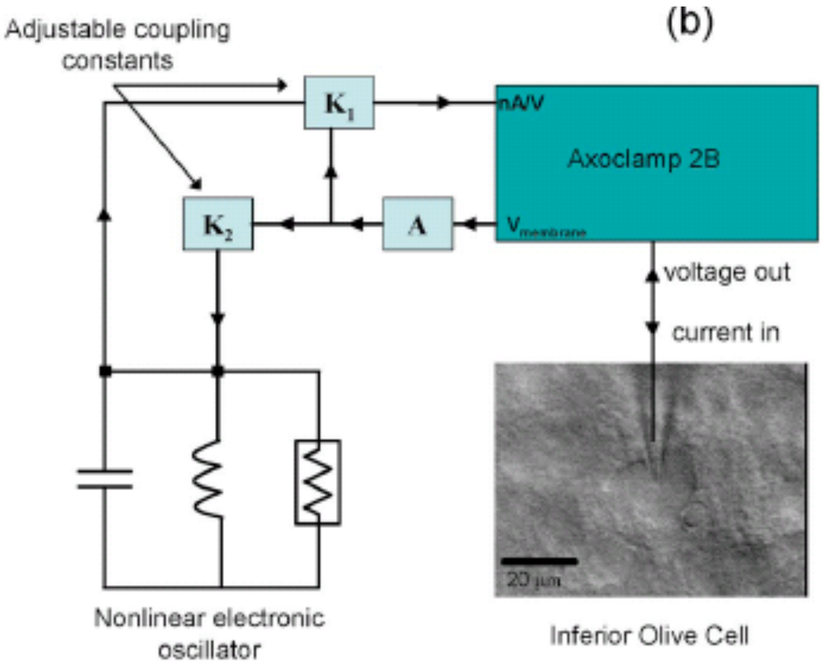




VOLUME 93, NUMBER 15 PHYSICAL REVIEW LETTERS week ending 8 OCTOBER 2004

Strong Coupling of Nonlinear Electronic and Biological Oscillators: Reaching the "Amplitude Death" Regime

I. Ozden,[1] S. Venkataramani,[1] M. A. Long,[3] B.W. Connors,[3] and A.V. Nurmikko[1,2,*]



Observed amplitude death in a coupled system of an electronic oscillator and a biological oscillator

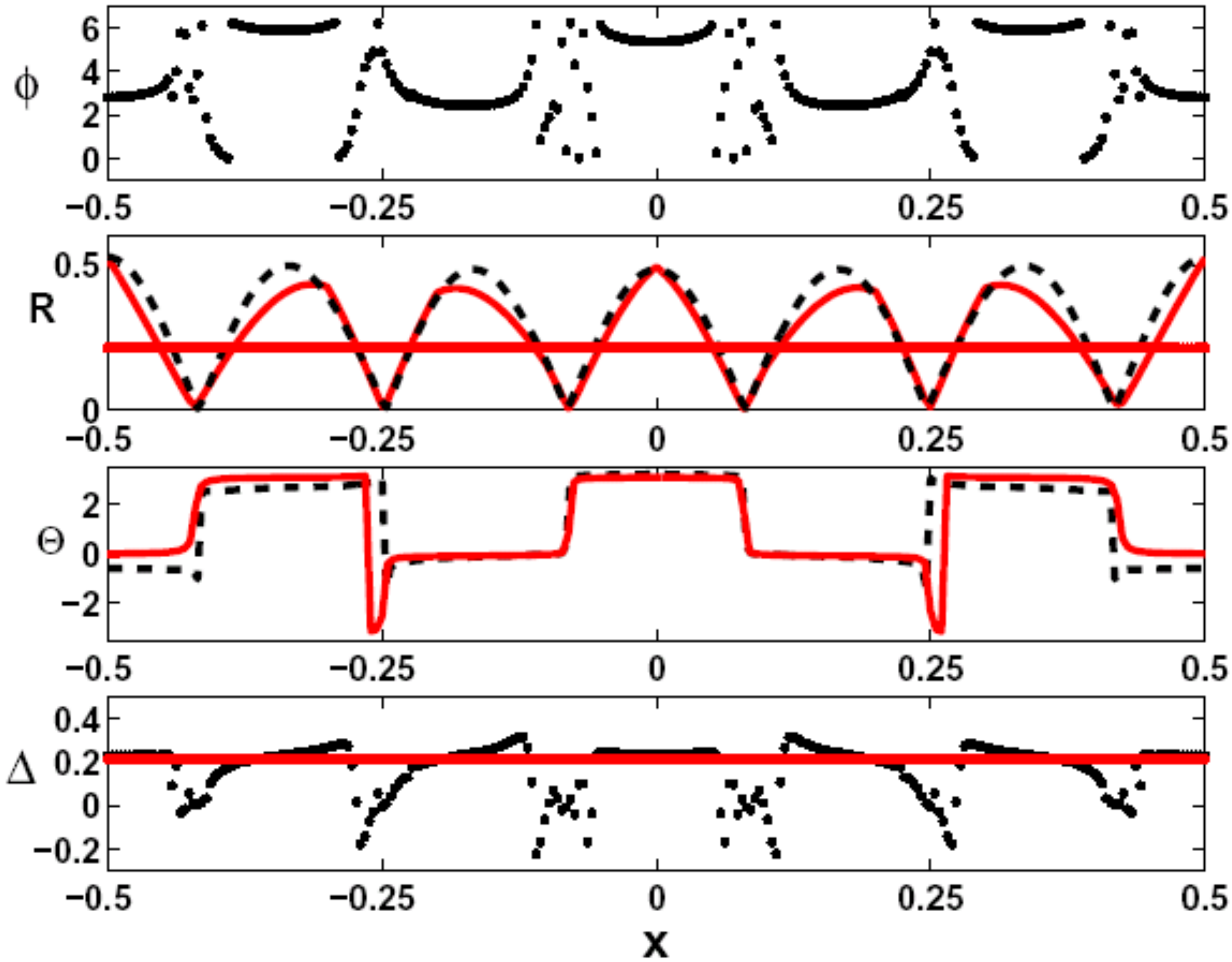
Non-local time delayed coupling

$$\psi(x,t) = r(x,t)e^{i\phi(x,t)}$$

Ignore amplitude variations

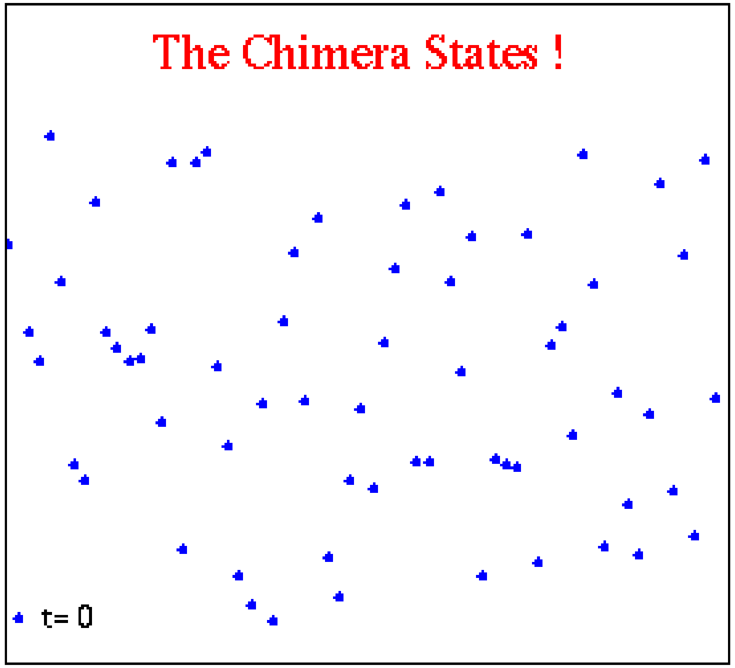
$$\boxed{\begin{aligned}\frac{\partial}{\partial t}\phi(x,t) = \omega - \int_{-\pi}^{\pi} & G(x-x') \\ & \times \sin\left[\phi(x,t) - \phi\left(x', t - \frac{|x-x'|}{v}\right) + \alpha\right] dx'\end{aligned}}$$

Do Chimera states exist in a time delayed system?

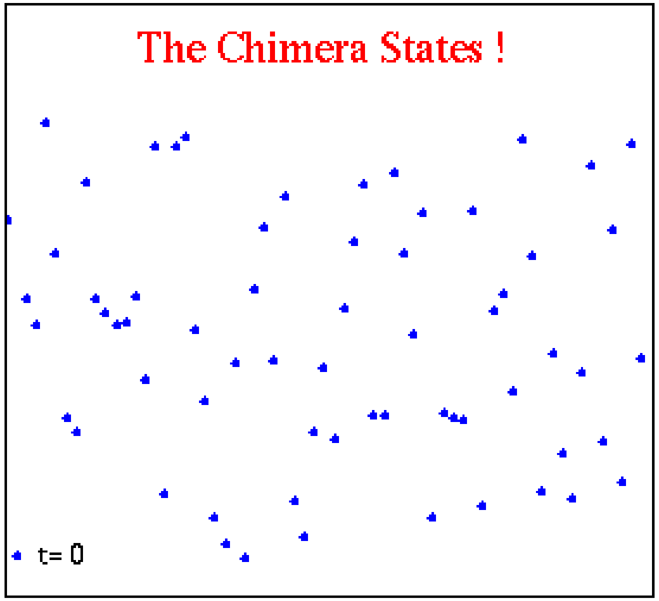


Sethia, Sen & Atay, PRL (2008)

Chimera states

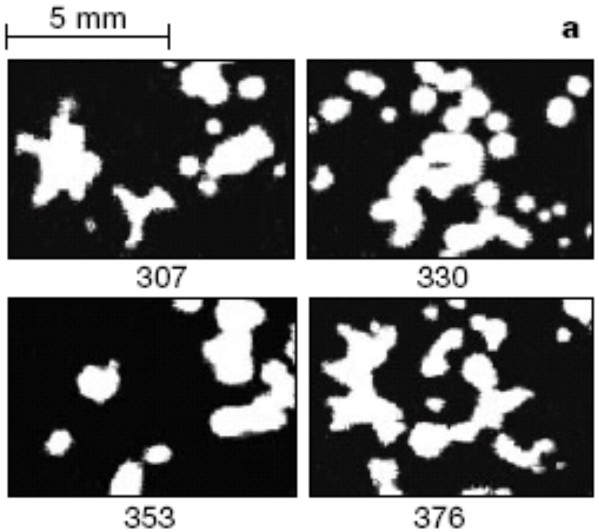


No delay



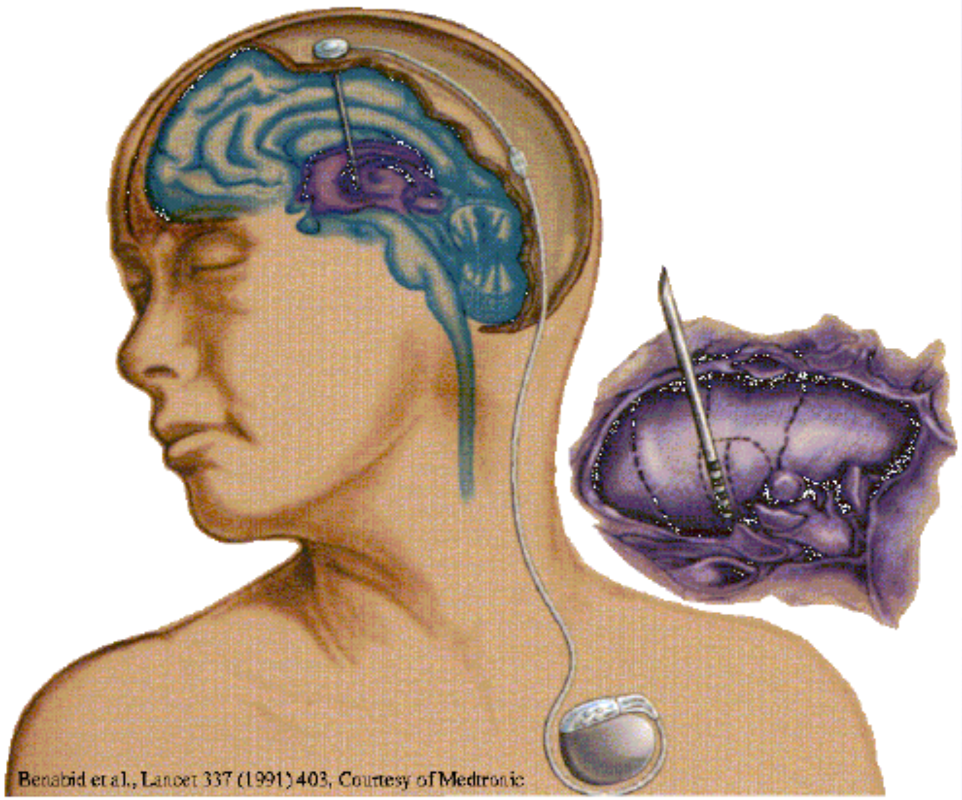
With delay

Irregular clusters of synchronized phase oscillations in BZ reactions



Vladimir K. Vanag, Lingfa Yang, Milos Dolnik, Anatol M. Zhabotinsky & Irving R. Epstein, Nature 406 (2000) 389

Deep Brain Stimulation

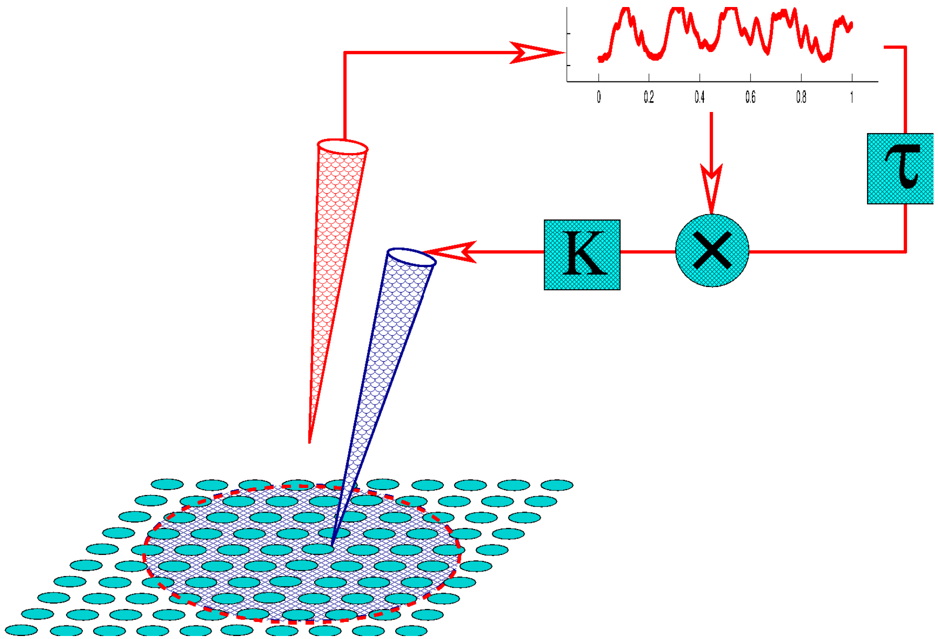


- strong synchronization of neuronal clusters may cause different disease symptoms like peripheral tremor (Morbus Parkinson) or epileptic seizures

Treatment:

- strong permanent pulsetrain stimulation signal
- suppress or over-activate neuronal activity
- may cause severe side effects

Stimulation with nonlinear delayed feedback



Basic Idea:
Desynchronize using a feedback signal

Stimulation signal $$S(t) = K\bar{Z}^2(t)\bar{Z}^*(t-\tau)$$

[3] O.V. Popovych, C. Hauptmann, and P.A. Tass, *Phys. Rev. Lett.* **94**, 164102 (2005)

Time delay helps in reducing the threshold for desynchronization

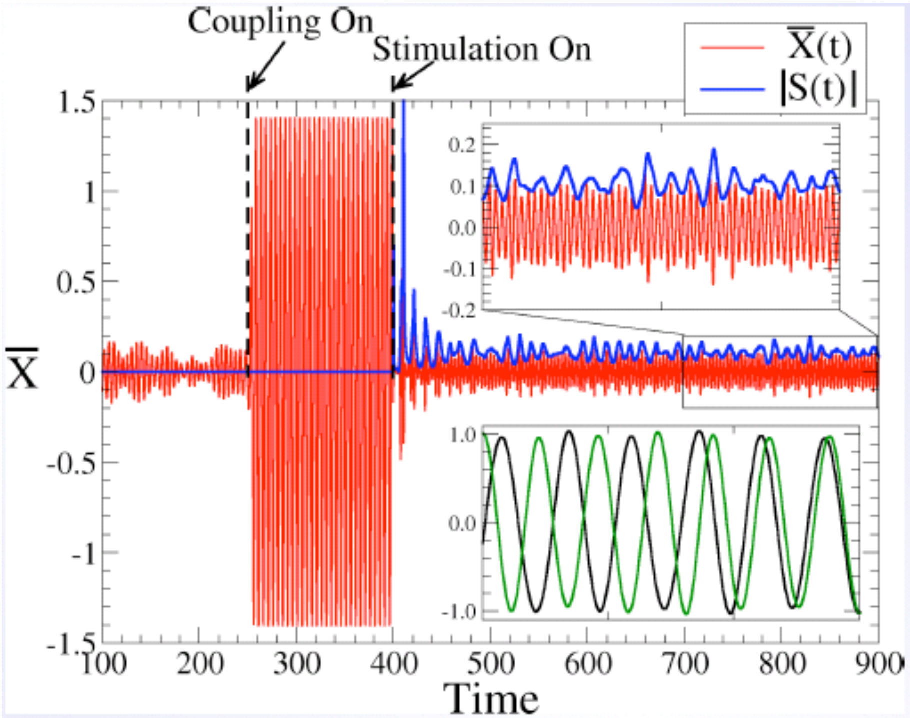
Model calculation using coupled limit cycle oscillator model

$$\dot{Z}_j(t) = (a_j + i\omega_j - |Z_j(t)|^2) Z_j(t) + C\overline{Z}(t) + \underbrace{K\overline{Z}^2(t)\overline{Z}^*(t-\tau)}_{\text{Stimulation Term}}$$

$$\overline{Z}(t) = \overline{X}(t) + i\overline{Y}(t) = \frac{1}{N}\sum_{j=1}^{N} Z_j(t)$$

$N = 100$, $a_j = 1.0$,
$\{\omega_j\}$ – Gaussian distributed:
mean $\Omega_0 = 2\pi/T$, $T = 5$
deviation $\sigma = 0.1$

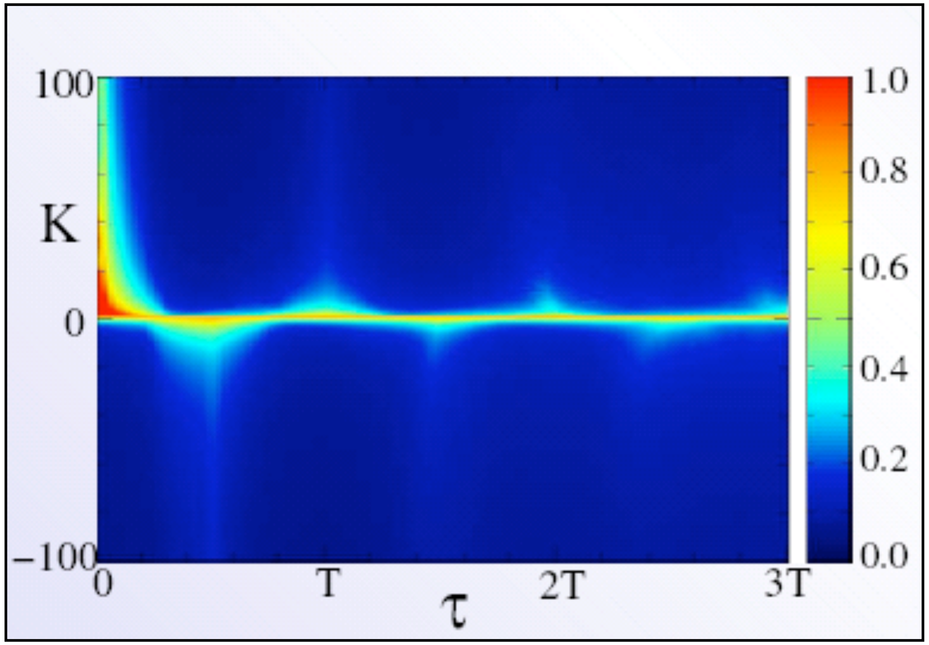
$C = 1$ for $t > 250$
$K = 150$ for $t > 400$
delay $\tau = 5.0 = T$



Effective desynchronization of coupled limit cycle oscillators

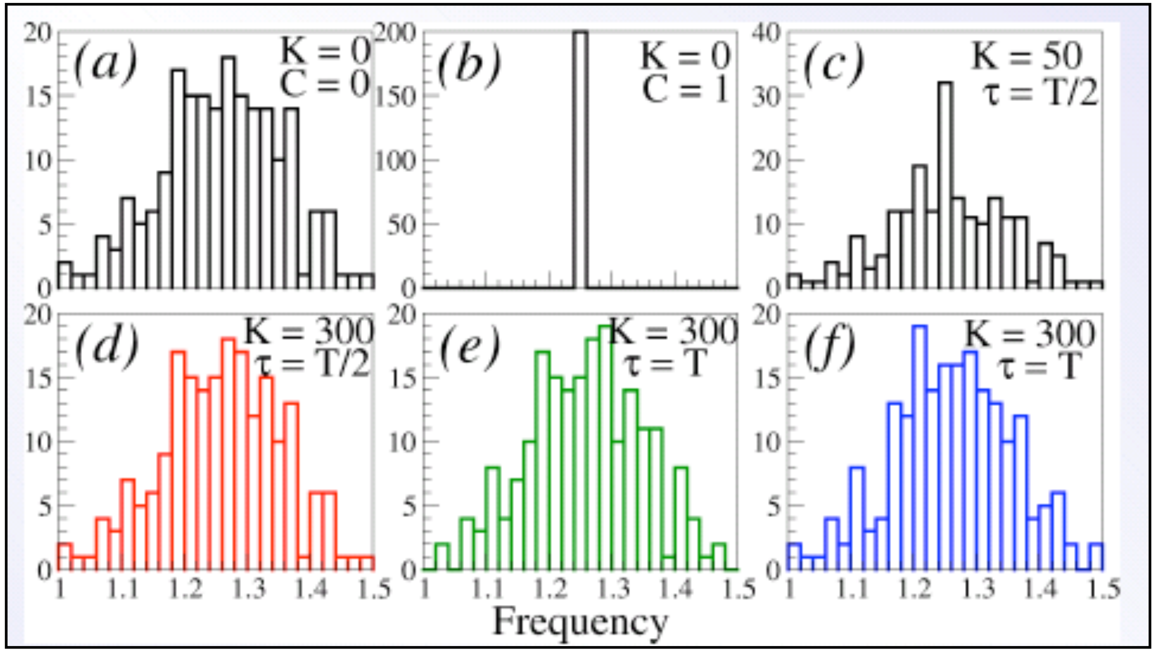
The averaged order parameter:

$$\langle R(t) \rangle := \left\langle \left| \frac{1}{N} \sum_{j=1}^{N} \frac{Z_j(t)}{|Z_j(t)|} \right| \right\rangle$$



Desynchronization mechanism

Stimulation restores individual frequencies of oscillators to natural frequencies



Concluding remarks:

- Coupled oscillator systems possess a rich variety of collective states which depend upon the coupling strength, nature of the coupling etc.

- ***Time delay in the coupling can have profound effects on the collective dynamics e.g. higher frequency states, amplitude death for identical oscillators, forbidden states etc***

- **Time delay can also enhance synchronization, facilitate desynchronization, induce bi-stability, influence chaos etc.**

- ***Useful paradigm for simulating and modeling many physical, chemical and biological systems***

- Collective dynamics of time delay coupled oscillator systems is an active and fertile area of research in applied mathematics, physics, biology, neuroscience.

- Vast potential for applications – communication, chaos control, simulation of turbulence in fluids, population dynamics ….. ….. list keeps growing

- Enormous opportunities for experimental studies as well e.g. nonlinear circuits, artificial neural nets, live studies of neurons. coupled lasers etc.

Acknowledgement

D.V. Ramana Reddy
G.L. Johnston
G. Sethia
Mitaxi Mehta
J. Kurths
F. Atay
S. Dana
H. Majumdar
Mitesh Patel
B.N. Buch