World of coupled oscillators

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Theme

• Coupled Oscillators provide a useful paradigm for the study of collective behavior of large complex systems

• A wonderful world to be in – full of interesting mathematical challenges and novel applications – physics, chemistry, biology, economics…..

• Still a very active area of research

• Lends itself easily to `hands-on style’ experiments!
Coupled Oscillators in the Natural World

• Walking, clapping, running…..
• Pacemaker cells in the heart
• Insulin secreting cells in the pancreas
• Neural networks in the brain and spinal cord
  -- control rhythmic behaviour like breathing …
• Groups of crickets, frogs in monsoon,
• Swarms of Fireflies

A common and striking occurrence is the emergence of a single rhythm – “synchrony”
QUESTIONS?

• How do coupled oscillators synchronize?

• Can one construct simple mathematical models to understand this phenomenon?

~ 1650

Observations and conjectures regarding Pendulum clocks

Huygens
Charles S. Peskin

Arthur T. Winfree

Mathematical Biologists

Pioneering work around 1970s
Charles S. Peskin (N.Y.U.) – circa 1975

- electrical circuit model for pacemaker cells
- capacitor in parallel with a resistor - constant input current - mimics firing of a pacemaker cell
- considered an array of identical oscillators - globally coupled (pulse coupling)

**TWO CONJECTURES**

- System would always eventually synchronize
- It would synchronize even if the oscillators are not quite identical
• **PESKIN PROVED HIS FIRST CONJECTURE FOR 2 OSCILLATORS** (ALSO FOUND AN OUT OF PHASE EQUILIBRIUM)

• **GENERAL PROOF FOR ARBITRARY NUMBER OF OSCILLATORS WAS OBTAINED 15 YRS LATER** *(STROGATZ & MIROLLO)*

• **ARTHUR T. WINFREE** (1966) - graduate student at Princeton

  • **MAJOR BREAKTHROUGH**
  • CONSIDERED SYSTEM OF COUPLED **LIMIT CYCLE OSCILLATORS**
  • **WEAK COUPLING APPROXIMATION**
  • CONSIDERED ONLY **PHASE VARIATIONS**
  • **GLOBAL COUPLING**

• **Y. Kuramoto** – developed the model further and made extensive use of it.
LIMIT CYCLE OSCILLATOR

\[ X'' + a (X^2 - 1)X' + X = 0, \quad a > 0 \]

"Isolated closed curve in phase space"
Van der Pol Oscillator

\[ i = C \frac{du_g}{dt} \]

\[ X = u_g \]
Belousov Zhabotinsky Reaction

Citric acid and bromate ions in a solution of sulfuric acid, and in the presence of a cerium catalyst.

\[ \dot{X} = a - X - \frac{4XY}{1+X^2} \]
\[ \dot{Y} = bX \left(1 - \frac{Y}{1+X^2}\right) \]

a=10, b=2
A SINGLE HOPF BIFURCATION OSCILLATOR

\[ Z'(t) = (a + i\omega - |Z(t)|^2)Z(t) \quad \rightarrow \quad \frac{d}{dt} \]

where \( Z = X + iY = r \exp(i\theta) \)

\[ \begin{align*}
    r' &= r(a - r^2) \\
    \theta' &= \omega
\end{align*} \]

Stewart – Landau Oscillator

\( \delta r' = a \delta r \)

Origin (r=0) stable for \( a \leq 0 \)
for \( a > 0 \) limit cycle osc.
Two Coupled Limit cycle Oscillators

\[
\dot{Z}_1(t) = (1 + i\omega_1 - |Z_1(t)|^2) Z_1(t) + K [Z_2(t) - Z_1(t)],
\]

\[
\dot{Z}_2(t) = (1 + i\omega_2 - |Z_2(t)|^2) Z_2(t) + K [Z_1(t) - Z_2(t)],
\]

\[
K = \text{coupling constant; } a=1
\]
In polar coordinates

\[
\begin{align*}
\dot{r}_1 &= r_1(1 - K - r_1^2) + K r_2 \cos[\theta_2 - \theta_1], \\
\dot{r}_2 &= r_2(1 - K - r_2^2) + K r_1 \cos[\theta_1 - \theta_2], \\
\dot{\theta}_1 &= \omega_1 + K \frac{r_2}{r_1} \sin[\theta_2 - \theta_1], \\
\dot{\theta}_2 &= \omega_2 + K \frac{r_1}{r_2} \sin[\theta_1 - \theta_2].
\end{align*}
\]

**Weak coupling approximation**: separation of time scales – short time – relaxation to limit cycle – long time phases interact \(-\) let \(r_1 \approx r_2 \approx \text{constant}\)
Identical oscillators: \( \omega_1 = \omega_2 \)

Define \( \phi = \theta_2 - \theta_1 \)

\[
\dot{\phi} = -2K \sin(\phi)
\]

Force tries to reduce phase difference

**EQUILIBRIA**

\( \phi = 0 \Rightarrow \theta_1 = \theta_2 \) symmetric state

\( \phi = \pi \Rightarrow \theta_1 = \theta_2 + \pi \) anti-symmetric state

**PHASE LOCKING** - “synchrony” is only a part of the story - “symmetry breaking” - general scenario
PHASE EQUILIBRIA and ANIMAL GAITS

In Phase
Out of Phase Synchronization in Human Walking / Running
4 OSCILLATORS

\[ \theta_1 = \theta_2 ; \theta_3 = \theta_4 ; \theta_1 = \theta_3 + \pi -- \text{rabbit, camel, horse} \]

\[ \theta_2 = \theta_1 + \pi / 4 ; \theta_3 = \theta_2 + \pi / 4 ; \theta_4 = \theta_3 + \pi / 4 ; \text{ -elephant} \]

\[ \theta_1 = \theta_2 = \theta_3 = \theta_4 -- \text{GAZELLE} \]
HORSE GAITS

walk

trot

gallop

Running speed
Three Oscillators

\[ \theta_1 = \theta_2 = \theta_3 \]

\[ \theta_1 = \theta_2 + \pi/3 ; \theta_2 = \theta_3 + \pi/3 ; \]

\[ \theta_1 = \theta_2 ; \; \theta_3 \text{ no relation - same frequency} \]

\[ \theta_1 = \theta_2 + \pi ; \; \theta_3 \text{ has twice the frequency} \]
Two out of synchrony and one twice as fast
• 6 OSCILLATORS -- INSECTS, COCKROACHES ETC.

• CENTIPEDE! – traveling wave

Courtesy: Dan Goldman
QUESTION: Coupled osc. Equilibria and Animal gaits - is this a mere coincidence or is there a deeper connection?

- Active area of research
- Central pattern generators (brain and spine)
- Group theoretic methods coupled with generalized Hopf bifurcations
- Clinical experiments
2 NON-IDENTICAL OSCILLATORS

\[
\dot{\phi} = \Delta - 2K \sin(\phi)
\]

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1 + K \frac{r_2}{r_1} \sin[\theta_2 - \theta_1], \\
\dot{\theta}_2 &= \omega_2 + K \frac{r_1}{r_2} \sin[\theta_1 - \theta_2].
\end{align*}
\]

where \( \Delta = |\omega_1 - \omega_2| \)

PHASE LOCKING ONLY IF \( \Delta \leq 2K \)

Then \( \dot{\theta} = < \omega > = \frac{(\omega_1 + \omega_2)}{2} \) Common frequency

FREQUENCY ENTRAINMENT
Two Phase Coupled Oscillators

The diagram illustrates the relationship between phase drift and coupling strength (K). The yellow region represents the phase locked state, while the red region indicates the phase drift state.
N coupled (phase only) oscillators

\[ \dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N \]

Frequencies given by a unimodal distribution function

\[ g(\omega) = g(-\omega) \]

“global coupling” - mean field approximation

Complex order parameter:

\[ r(t) \quad \text{- measure of phase coherence} \]

\[ r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \quad \psi(t) \quad \text{- average phase} \]
Kuramoto solved the equation exactly for $r = \text{constant}$ and obtained the threshold condition for synchrony $K \geq K_c$.

\[
\dot{\theta}_i = \omega_i + Kr \sin(\psi - \theta_i), \quad i = 1, \ldots, N.
\]

$$K_c = \frac{2}{\pi g(0)}$$

$r = 1$ – synchrony

$r = 0$ – phase drift
\[ r = \sqrt{1 - \frac{K_c}{K}} \quad \text{for} \quad g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)} \]

“Second order phase transition”

Near onset
\[ r \approx \sqrt{\frac{16}{\pi K_c^3} \sqrt{\frac{\mu}{-g''(0)}}} \]

Supercritical bifurcation for \[ g''(0) < 0 \]

Synchronization in Fire Flies
Synchronization in fireflies

• S. Strogatz - "From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators"

Strong Coupling Limit: Amplitude effects

\[
\begin{align*}
\dot{Z}_1(t) &= (1 + i\omega_1 - |Z_1(t)|^2) Z_1(t) + K [Z_2(t) - Z_1(t)], \\
\dot{Z}_2(t) &= (1 + i\omega_2 - |Z_2(t)|^2) Z_2(t) + K [Z_1(t) - Z_2(t)],
\end{align*}
\]

\[|Z_1| = |Z_2| = 0\] is an equilibrium solution

Stability of the origin?

\[
\lambda^2 - 2(a + i\bar{\omega})\lambda + (b_1 + ib_2) + c = 0
\]

\[
a = 1 - K, \quad b_1 = a^2 - \bar{\omega}^2 + \Delta^2 / 4, \quad b_2 = 2a\bar{\omega}
\]

\[
c = -K^2. \quad \text{Origin stable if} \quad Re(\lambda) < 0.
\]
Marginal Stability Curve

Substitute $\lambda = \alpha + i\beta$ in characteristic equation and solve it for $\alpha = 0$

This yields $\beta = \bar{\omega}$

And the conditions: $K = 1, \bar{K} = \gamma(\Delta) = \frac{1}{2}(1 + \frac{\Delta^2}{4})$
Two Amplitude Coupled Oscillators

![Graph showing the regions of phase drift, amplitude death, and phase locked for two amplitude coupled oscillators.](image-url)
Physical picture of amplitude death

(Strong coupling limit)

Two oscillators

Each oscillator pulls the other off its limit cycle and they both collapse into the origin $r = 0$ -- **AMPLITUDE DEATH**

**Happens for $K$ large and $\Delta$ large**
EXAMPLES OF AMPLITUDE DEATH

• **CHEMICAL OSCILLATIONS** - BZ REACTIONS
  (coupled stirred tank reactors - Bar Eli effect)

• **POPULATION DYNAMICS**
  Two sites with same predator prey mechanism can have
  oscillatory behaviour. If species from one site can move
  to another at appropriate rate (appropriate coupling
  strength) the two sites may become stable (stop oscillating)

• **ORGAN PIPES**
\[ \dot{z}_j = z_j (1 - |z_j|^2 + i \omega_j) + \frac{K}{N} \sum_{i=1}^{N} (z_i - z_j) \]

R is the order parameter
Large Number of Amplitude Coupled Oscillators
So far we have looked at systems with “global” coupling – mean field coupling

What about other forms of coupling?

Short range interactions (nearest neighbour)?

Non-local coupling?

Time delayed coupling
EXTENSION TO SYSTEMS WITH SHORT RANGE INTERACTIONS

• Nearest neighbour coupling

• Limit of very large N – chain of identical oscillators

\[ \frac{\partial \psi_j}{\partial t} = (1 + i \omega_0 - |\psi_j|^2)\psi_j + K[\psi_{j+1} - \psi_j] + K[\psi_{j-1} - \psi_j], \]

In the continuum limit set \( \psi_j = \psi(ja) \)

Let \( a \to 0 \); \( ja \to x \)

\[ \frac{\partial \psi(x,t)}{\partial t} = (1 + i \omega_0 - |\psi(x,t)|^2)\psi(x,t) + K \frac{\partial^2 \psi^2(x,t)}{\partial x^2} \]

Complex Ginzburg Landau Eqn
Non-local coupling

\[ \frac{\partial \psi_j(t)}{\partial t} = (1 + i\omega_0 - |\psi_j|^2)\psi_j + K \sum_{i=1}^{N} G(j-i)(\psi_i - \psi_j) \]

Continuum limit:

\[ \frac{\partial \psi(x,t)}{\partial t} = (1 + i\omega_0 - |\psi(x,t)|^2)\psi(x,t) \]

\[ + K \int_{-L}^{L} G(x-x') [\psi(x',t) - \psi(x,t)] \, dx' \]

Non-local CGLE
Weak coupling limit

\[ \psi(x, t) = r(x, t)e^{i\phi(x, t)} \]

Ignore amplitude variations

\[ \frac{\partial \phi}{\partial t} = \omega - \int_{-\pi}^{\pi} G(x - x') \sin[\phi(x, t) - \phi(x', t) + \alpha] dx'. \]

``Ring of identical phase oscillators with non-local coupling``

Compare with

\[ \dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \]

\[ G(y) = \frac{\kappa}{2} \exp(-\kappa|y|) \]

Kuramoto and Battogtokh,
“Novel” collective state

Simultaneous existence of coherent and incoherent states

$k = 4.0$, $\alpha = 1.45$, $N = 256$ oscillators.

“Chimera” state
Understanding the Chimera state

Define a rotating frame with frequency $\Omega$

Relative phase in that frame

$$\theta = \phi - \Omega t$$

$$R(x, t)e^{i\Theta(x, t)} = \int_{-\pi}^{\pi} G(x - x')e^{i\theta(x', t)} dx'.$$

Complex order parameter

$$\frac{\partial \theta}{\partial t} = \omega - \Omega - R \sin[\theta - \Theta + \alpha]$$

Look for stationary solutions in which $R$ and $\Theta$ are space dependent

$$\omega - \Omega = R(x) \sin[\theta^* - \Theta(x) + \alpha]$$

Oscillators with $R(x) \geq |\omega - \Omega|$ Fixed pt. $\theta^*$

$R(x) < |\omega - \Omega|$ drift around the phase circle monotonically.
Time delayed coupling?

Time delay is ubiquitous in real systems due to finite propagation speed of signals, finite reaction times of Chemical reactions, finite response time of synapses etc.

WHAT HAPPENS TO THE COLLECTIVE DYNAMICS OF COUPLED SYSTEMS IN THE PRESENCE OF TIME DELAY?
SIMPLE TIME DELAYED MODEL

\[ \dot{Z}_1(t) = (1 + i\omega_1 - |Z_1(t)|^2)Z_1(t) + K[Z_2(t - \tau) - Z_1(t)], \]
\[ \dot{Z}_2(t) = (1 + i\omega_2 - |Z_2(t)|^2)Z_2(t) + K[Z_1(t - \tau) - Z_2(t)], \]

Weak coupling limit

\[
\begin{align*}
\dot{\theta}_1 &= \omega_1 + K \sin[\theta_2(t - \tau) - \theta_1], \\
\dot{\theta}_2 &= \omega_2 + K \sin[\theta_1(t - \tau) - \theta_2].
\end{align*}
\]

**Phase locked solution:**

\[
\dot{\theta}_j = \Omega t + \alpha
\]

\[
\Omega = \bar{\omega} - 2K \sin(\Omega \tau)
\]

\[
\bar{\omega} = \frac{\omega_1 + \omega_2}{2}
\]

- Multiple frequency states
- Frequency suppression

\[
\Omega \approx \frac{\bar{\omega}}{1 + 2K \tau}
\]
Early work by Schuster et al

- **Multiple Frequencies:**

- **Frequency suppression**


**Kuramoto model**

\[
\frac{d\phi_i(t)}{dt} = \omega_0 + K \sum_j \sin[\phi_j(t - \tau) - \phi_i(t)] + \eta_i
\]
Strong Coupling Limit:

Linear stability analysis of the origin $Z=0$

Eigenvalue equation:

$$(a - \lambda + i\omega_1)(a - \lambda + i\omega_2) - K^2 e^{-2\lambda \tau} = 0$$

For $\tau = 0$ detailed analysis by D.G. Aronson, G.B. Ermentrout and N. Kopell, Physics 41 D (1990) 403

$a = 1 - K$
Two Coupled Oscillators with Delay

Identical Oscillators can DIE!

Geometric Interpretation of delay induced death in identical oscillators

The current state $P(t)$ is pulled towards the retarded state $Q(t-\tau)$ of the other oscillator and vice-versa. For appropriate values of $K$ and time delay both oscillations will spiral inwards and die out.
• Existence of death islands in $K - \tau$ space

Size, shape vary with $N$ and $\omega$
- Existence of multiple death islands
• Existence of higher frequency states and their stability
Experimental verification carried out on coupled nonlinear circuits (Reddy, Sen, Johnston, PRL, 85 (2000) 3381)

\[ \ddot{V}_i + g(V_i)\dot{V}_i + \omega_i^2 V_i = K_i[\dot{V}_j(t - \tau) - \dot{V}_i(t)] \]

FIG. 2. The V-I characteristics of the nonlinear component \( R_N \). The continuous line is a polynomial fit of the experimental points.
• Death state confirmed
• In-phase and out-of-phase oscillations seen
• Existence of death islands and their multiple connectedness.
• IN-PHASE AND ANTI-PHASE LOCKED STATES

FIG. 6. (a) Coexistence of in-phase- and anti-phase-locked states and (b) suppression of the phase-locked states as $\tau$ is increased for $K = 1000 \, s^{-1}$ and $\omega = 837 \, s^{-1}$. 
Time delay effects in a living coupled oscillator system

(Takamatsu et al, PRL 85 (2000) 2026)

Experiments with plasmodium of slime mold

- contraction/relaxation states
- time delay and coupling controlled by size of tube
- observed in-phase/anti-phase oscillations
Observed amplitude death in a coupled system of an electronic oscillator and a biological oscillator
Non-local time delayed coupling

\[ \psi(x, t) = r(x, t)e^{i\phi(x, t)} \]

Ignore amplitude variations

\[ \frac{\partial}{\partial t} \phi(x, t) = \omega - \int_{-\pi}^{\pi} G(x - x') \times \sin \left[ \phi(x, t) - \phi \left( x', t - \frac{|x - x'|}{v} \right) + \alpha \right] dx' \]

Do Chimera states exist in a time delayed system?
Sethia, Sen & Atay, PRL (2008)
Chimera states

The Chimera States!

No delay

The Chimera States!

With delay
Irregular clusters of synchronized phase oscillations in BZ reactions

Deep Brain Stimulation

• strong synchronization of neuronal clusters may cause different disease symptoms like peripheral tremor (Morbus Parkinson) or epileptic seizures

Treatment:
• strong permanent pulsetrain stimulation signal
• suppress or over-activate neuronal activity
• may cause severe side effects
Stimulation with nonlinear delayed feedback

Basic Idea:
Desynchronize using a feedback signal

\[ S(t) = K \tilde{Z}^2(t) \tilde{Z}^*(t - \tau) \]


Time delay helps in reducing the threshold for desynchronization
Model calculation using coupled limit cycle oscillator model

\[ \dot{Z}_j(t) = (a_j + i\omega_j - |Z_j(t)|^2)Z_j(t) + C\overline{Z}(t) + K\overline{Z}^2(t)\overline{Z}^*(t - \tau) \]

Stimulation Term

\[ \overline{Z}(t) = \overline{X}(t) + i\overline{Y}(t) = \frac{1}{N} \sum_{j=1}^{N} Z_j(t) \]

\[ N = 100, \ a_j = 1.0, \ \{\omega_j\} \text{ - Gaussian distributed:} \]
\[ \text{mean} \ \Omega_0 = 2\pi/T, \ T = 5 \]
\[ \text{deviation} \ \sigma = 0.1 \]

\[ C' = 1 \text{ for } t > 250 \]
\[ K = 150 \text{ for } t > 400 \]
\[ \text{delay} \ \tau = 5.0 = T \]
Effective desynchronization of coupled limit cycle oscillators

The averaged order parameter:

\[ \langle R(t) \rangle := \left\langle \left| \frac{1}{N} \sum_{j=1}^{N} \frac{Z_j(t)}{|Z_j(t)|} \right| \right\rangle \]
Desynchronization mechanism

Stimulation restores individual frequencies of oscillators to natural frequencies
Concluding remarks:

- Coupled oscillator systems possess a rich variety of collective states which depend upon the coupling strength, nature of the coupling etc.

- Time delay in the coupling can have profound effects on the collective dynamics e.g. higher frequency states, amplitude death for identical oscillators, forbidden states etc

- Time delay can also enhance synchronization, facilitate desynchronization, induce bi-stability, influence chaos etc.

- Useful paradigm for simulating and modeling many physical, chemical and biological systems
• **Collective dynamics of time delay coupled oscillator systems**
  is an active and fertile area of research in applied mathematics,
  physics, biology, neuroscience.

• **Vast potential for applications – communication, chaos control,**
  **simulation of turbulence in fluids, population dynamics …..**
  ….. list keeps growing

• **Enormous opportunities for experimental studies as well e.g.**
  **nonlinear circuits, artificial neural nets, live studies of neurons,**
  **coupled lasers etc.**
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